Chapter 3

Basic Morse Theory

The main goal of this chapter is to show how to construct a CW-complex that is homotopy equivalent to a given smooth manifold M using some special functions on M called "Morse" functions (Theorem 3.28). The CW-homology of the resulting CW-complex is isomorphic to the singular homology of M by Theorem 2.15, and hence it is independent of the choice of the Morse function used to build the CW-complex. As a consequence we derive the Morse inequalities. The last section of this chapter is an introduction to Morse-Bott functions.

3.1 Morse functions

If p is a point on a smooth manifold M of dimension m, we denote by T_pM the tangent space of M at p. Recall that a tangent vector $V \in T_pM$ is an equivalence class of curves $\gamma : (-\varepsilon, \varepsilon) \to M$ for some $\varepsilon > 0$ with $\gamma(0) = p$. Two curves are equivalent if they have the same velocity at 0, i.e.

$$V = \left.\frac{d\gamma}{dt}\right|_{t=0}.$$

We will also write $V = [\gamma]$.

A **critical point** of a smooth function $f : M \to \mathbb{R}$ is a point p at which the differential

$$df_p : T_pM \to T_{f(p)}\mathbb{R} \approx \mathbb{R}$$

vanishes. Recall that if $V = [\gamma] \in T_pM$, then

$$df_p(V) = \left.\frac{d}{dt} f(\gamma(t))\right|_{t=0} \quad \text{or} \quad df_p([\gamma]) = [f \circ \gamma].$$