Note that this allows us to find the homology groups of $\mathbb{C}P^n$ by simply counting the number of critical points of each index. We will see that a similar result holds for complex Grassmann manifolds in Chapter 8.

Definition 3.41 *If $f : M \to \mathbb{R}$ is a Morse function such that $M_t(f) = P_t(M)$, then f is called a* ***perfect Morse function****.*

Note that if a manifold admits a perfect Morse function, then its homology doesn't have any torsion. For more details about perfect Morse functions as well as examples where the Morse inequalities are not sharp see [125].

The next theorem is an important result that follows as an easy consequence of Theorem 3.33.

Theorem 3.42 *Let M be a compact manifold of odd dimension, then the Euler characteristic is zero, i.e. $\mathcal{X}(M) = 0$.*

Proof:

Let $f : M \to \mathbb{R}$ be a Morse function, and assume that the dimension m of the manifold M is odd. Since $\nu_k(f) = \nu_{m-k}(-f)$ we have the following.

$$\begin{aligned}
\mathcal{X}(M) &= \sum_{k=0}^{m}(-1)^k \nu_k(f) \\
&= \sum_{k=0}^{m}(-1)^k \nu_{m-k}(-f) \\
&= (-1)^m \sum_{k=0}^{m}(-1)^{m-k} \nu_{m-k}(-f) \\
&= (-1)^m \sum_{k=0}^{m}(-1)^k \nu_k(-f) \\
&= (-1)^m \mathcal{X}(M)
\end{aligned}$$

Hence, $\mathcal{X}(M) = 0$ if m is odd.

□

We will now prove the Morse inequalities. We will see that Theorem 3.33 and Theorem 3.36 are both easy consequences of Theorem 3.28. However, it should be noted that Morse proved his inequalities in the 1930's before Theorem 3.28 was known [113]. Morse's inequalities provided the first deep relationship between the critical points of a real valued function on a smooth manifold M and the topology of M. For an exposition of Morse's approach to