

An Overview of Morse-Bott Homology

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Multicomplexes

Morse functions

1. The **Hessian** $H_p(f)$ of a smooth function $f : M \rightarrow \mathbb{R}$ at a critical point $p \in M$ is a symmetric bilinear map $H_p(f) : T_pM \times T_pM \rightarrow \mathbb{R}$ whose matrix in local coordinates $\phi(x) = (x_1, \dots, x_m)$ is given by

$$M_p(f) = \left(\frac{\partial^2 (f \circ \phi^{-1})}{\partial x_i \partial x_j} \phi(p) \right).$$

2. The dimension of the subspace of T_pM on which $H_p(f)$ is negative definite is called the **index** of p , i.e. the number of negative eigenvalues of $M_p(f)$, and is denote by λ_p .
3. The critical point p is said to be **non-degenerate** if and only if the Hessian $H_p(f)$ is non-degenerate.
4. A **Morse function** $f : M \rightarrow \mathbb{R}$ on a smooth manifold M is a smooth function whose critical points are all non-degenerate.

Stable and unstable manifolds

Let $p \in M$ be a critical point of a smooth function $f : M \rightarrow \mathbb{R}$ on a smooth Riemannian manifold (M, g) of dimension $m < \infty$, and let $\varphi : \mathbb{R} \times M \rightarrow M$ be the 1-parameter family of diffeomorphisms determined by $-\nabla f$. The **stable manifold** of p is

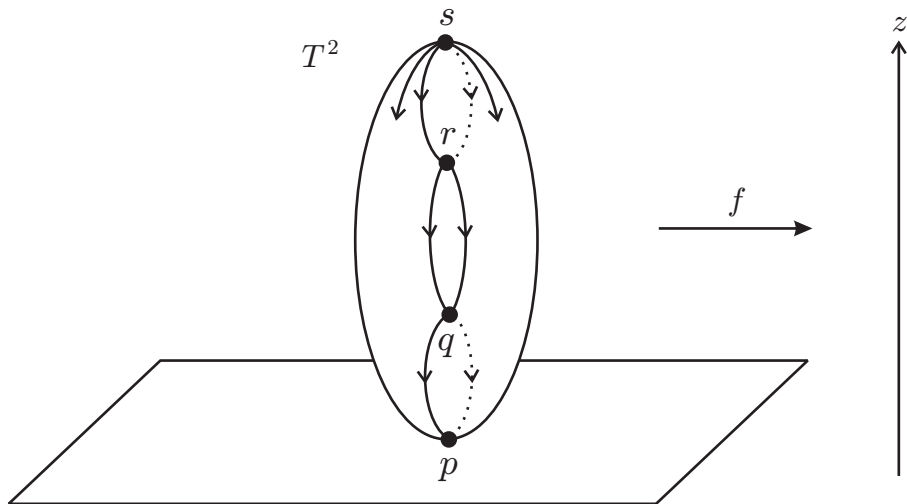
$$W^s(p) = \{x \in M \mid \lim_{t \rightarrow \infty} \varphi_t(x) = p\}$$

and the **unstable manifold** of p is

$$W^u(p) = \{x \in M \mid \lim_{t \rightarrow -\infty} \varphi_t(x) = p\}.$$

The Stable/Unstable Manifold Theorem: If p is a nondegenerate critical point, then the stable manifold $W^s(p)$ is a smoothly embedded open disk of dimension $m - \lambda_p$ and the unstable manifold $W^u(p)$ is a smoothly embedded open disk of dimension λ_p .

Stable and unstable manifolds on T^2



Morse-Smale transversality

A pair (f, g) is called **Morse-Smale** if and only if all the stable and unstable manifolds intersect transversally, i.e. $W^u(q) \pitchfork W^s(p)$ for all $p, q \in Cr(f)$.

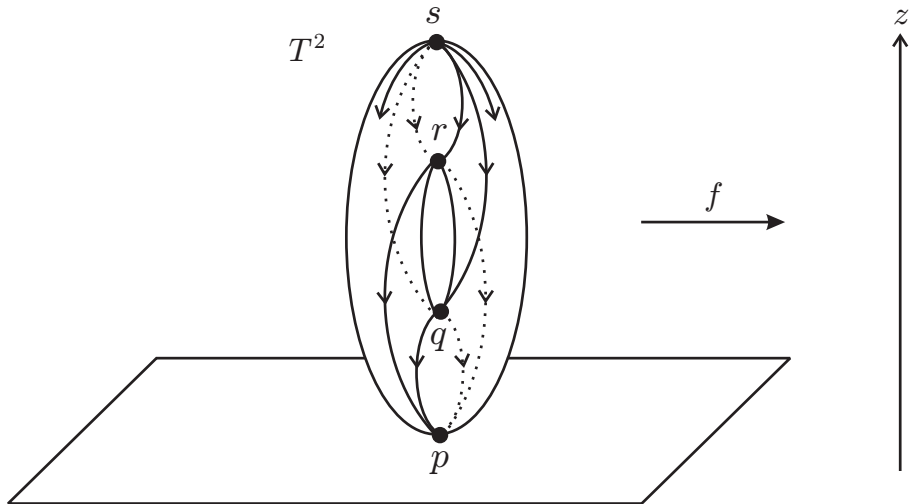
If $W^u(q) \cap W^s(p) \neq \emptyset$, then this condition implies that $W^u(q) \cap W^s(p)$ is a manifold of dimension $\lambda_q - \lambda_p$ and the **moduli space**

$$\mathcal{M}(q, p) = (W^u(q) \cap W^s(p)) / \mathbb{R}$$

is a manifold of dimension $\lambda_q - \lambda_p - 1$.

Note: The dimension of M **does not affect** the dimension of the moduli space $\mathcal{M}(q, p)$.

A Morse-Smale function on T^2 (tilted)



The Morse-Smale-Witten chain complex

Let $f : M \rightarrow \mathbb{R}$ be a Morse-Smale function on a compact smooth Riemannian manifold M of dimension $m < \infty$, and assume that orientations for the unstable manifolds of f have been chosen. Let $C_k(f)$ be the free abelian group generated by the critical points of index k , and let

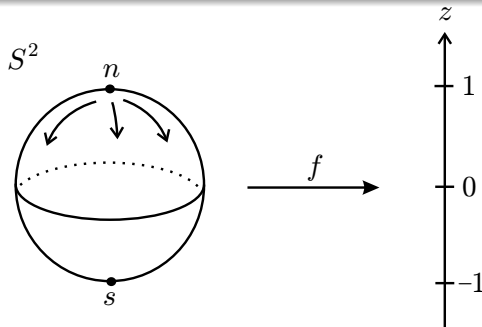
$$C_*(f) = \bigoplus_{k=0}^m C_k(f).$$

Define a homomorphism $\partial_k : C_k(f) \rightarrow C_{k-1}(f)$ by

$$\partial_k(q) = \sum_{p \in \text{Cr}_{k-1}(f)} n(q, p) p$$

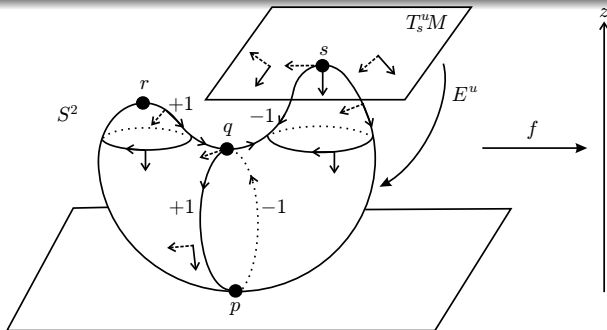
where $n(q, p)$ is the number of gradient flow lines from q to p counted with sign. The pair $(C_*(f), \partial_*)$ is called the **Morse-Smale-Witten chain complex** of f .

The height function on the 2-sphere



$$\begin{array}{ccccccc}
 C_2(f) & \xrightarrow{\partial_2} & C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\
 \uparrow \approx & & \uparrow \approx & & \uparrow \approx & & \\
 \langle n \rangle & \xrightarrow{\partial_2} & \langle 0 \rangle & \xrightarrow{\partial_1} & \langle s \rangle & \longrightarrow & 0
 \end{array}$$

The height function on a deformed 2-sphere



$$\begin{array}{ccccccc}
 C_2(f) & \xrightarrow{\partial_2} & C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\
 \uparrow \approx & & \uparrow \approx & & \uparrow \approx & & \\
 \langle r, s \rangle & \xrightarrow{\partial_2} & \langle q \rangle & \xrightarrow{\partial_1} & \langle p \rangle & \longrightarrow & 0
 \end{array}$$

References for Morse homology

- ▶ Audin, Michèle and Damian, Mihai. **Morse theory and Floer homology**. Translated from the 2010 French original by Reinie Ern  . Universitext. Springer, 2014.
- ▶ Augustin Banyaga and David Hurtubise, **Lectures on Morse homology**, Kluwer Texts in the Mathematical Sciences 29, Kluwer Academic Publishers Group, 2004.
- ▶ John Milnor, **Lectures on the h-cobordism theorem**, Princeton University Press, 1965.
- ▶ Matthias Schwarz, **Morse homology**, Progress in Mathematics 111, Birkh  user, 1993.
- ▶ Edward Witten, *Supersymmetry and Morse theory*, J. Differential Geom. 17 (1982), no. 4, 661–692.

Morse-Bott functions

Definition

A smooth function $f : M \rightarrow \mathbb{R}$ on a smooth manifold M is called a **Morse-Bott function** if and only if $\text{Cr}(f)$ is a disjoint union of connected submanifolds, and for each connected submanifold $B \subseteq \text{Cr}(f)$ the normal Hessian is non-degenerate for all $p \in B$.

Lemma (Morse-Bott Lemma)

Let $f : M \rightarrow \mathbb{R}$ be a Morse-Bott function, and let B be a critical submanifold. For any $p \in B$ there is a local chart of M around p and a local splitting of the normal bundle

$\nu_*(B) = \nu_*^+(B) \oplus \nu_*^-(B)$ identifying a point $x \in M$ in its domain with $(u, v, w) \in B \oplus \nu_*^+(B) \oplus \nu_*^-(B)$ such that within this chart f assumes the form

$$f(x) = f(u, v, w) = f(B) + |v|^2 - |w|^2.$$

Stable/Unstable manifolds

Pick a Riemannian metric g on M , and let φ_t be the flow of $-\nabla f$. For $p \in Cr(f)$ the **stable manifold** $W^s(p)$ and the **unstable manifold** $W^u(p)$ are defined as follows.

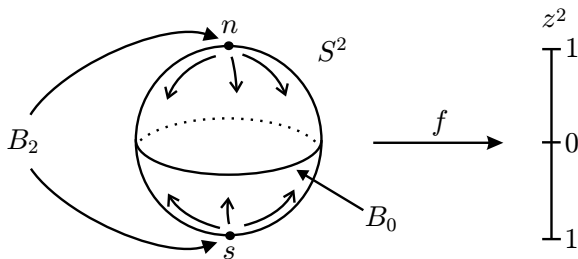
$$\begin{aligned} W^s(p) &= \{x \in M \mid \lim_{t \rightarrow \infty} \varphi_t(x) = p\} && (f \text{ decreases to } p) \\ W^u(p) &= \{x \in M \mid \lim_{t \rightarrow -\infty} \varphi_t(x) = p\} && (f \text{ increases to } p) \end{aligned}$$

Definition

If $f : M \rightarrow \mathbb{R}$ is a Morse-Bott function, then the stable and unstable manifolds of a critical submanifold B are defined to be

$$\begin{aligned} W^s(B) &= \bigcup_{p \in B} W^s(p) \\ W^u(B) &= \bigcup_{p \in B} W^u(p). \end{aligned}$$

A Morse-Bott function on the 2-sphere



Can we construct a chain complex for this function? a spectral sequence? a multicomplex?

Generic perturbations

Theorem (Morse 1932)

Let M be a finite dimensional smooth manifold. Given any smooth function $f : M \rightarrow \mathbb{R}$ and any $\varepsilon > 0$, there is a Morse function $g : M \rightarrow \mathbb{R}$ such that $\sup\{|f(x) - g(x)| \mid x \in M\} < \varepsilon$.

Theorem

Let M be a finite dimensional compact smooth manifold. The space of all C^r Morse functions on M is an open dense subspace of $C^r(M, \mathbb{R})$ for any $2 \leq r \leq \infty$ where $C^r(M, \mathbb{R})$ denotes the space of all C^r functions on M with the C^r topology.

Why not just perturb the Morse-Bott function $f : M \rightarrow \mathbb{R}$ to a Morse function?

The Chern-Simons functional

Let $P \rightarrow N$ be a (trivial) principal $SU(2)$ -bundle over an oriented closed 3-manifold N , and let $\mathcal{A}$ be the space of connections on P . Define $CS : \mathcal{A} \rightarrow \mathbb{R}$ by

$$CS(A) = \frac{1}{4\pi^2} \int_M \text{tr} \left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A \right).$$

The above functional descends to a function $cs : \mathcal{A}/\mathcal{G} \rightarrow \mathbb{R}/\mathbb{Z}$ whose critical points are gauge equivalence classes of flat connections. Extending everything to $P \times \mathbb{R} \rightarrow N \times \mathbb{R}$, the gradient flow equation becomes the instanton equation

$$F + *F = 0,$$

where F denotes the curvature and $*$ is the Hodge star operator.

Instanton homology

Andreas Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. 118 (1988), no. 2, 215–240.

Theorem. (Floer 1988) When N is a homology 3-sphere the Chern-Simons functional can be **perturbed so that it has discrete critical points** and defines $\mathbb{Z}_8$ -graded homology groups $I_*(N)$ analogous to the Morse homology groups.

The Euler characteristic of $I_*(N)$ is twice the Casson invariant.

Generalizations: Donaldson polynomials for 4-manifolds with boundary, knot homology groups

Monopole Floer homology for three and four manifolds

Modern approaches use the Chern–Simons–Dirac functional $\mathcal{L}(B, \Psi)$, defined on a spin^c connection B and a section Ψ of a corresponding spin bundle.

The functional $\mathcal{L}$ is a circle-value function (similar to Novikov homology), and the construction of the chain complex uses local coefficients on a manifold with boundary. The “gradient” flow is given by the **Seibert–Witten equations**.

Peter Kronheimer and Thomasz Mrowka. **Monopoles and three-manifolds**. New Mathematical Monographs, 10. Cambridge University Press, Cambridge, 2007. xii+796pp.

The symplectic action functional

Let (M, ω) be a closed symplectic manifold and $S^1 = \mathbb{R}/\mathbb{Z}$. A time-dependent Hamiltonian $H : M \times S^1 \rightarrow \mathbb{R}$ determines a time-dependent vector field X_H by

$$\omega(X_H(x, t), v) = v(H)(x, t) \text{ for } v \in T_x M.$$

Let $\mathcal{L}(M)$ be the free contractible smooth loops on M and

$$\tilde{\mathcal{L}}(M) = \{(x, u) | x \in \mathcal{L}(M), u : D^2 \rightarrow M \text{ such that } u(e^{2\pi i t}) = x(t)\} / \sim$$

its universal cover with covering group $\pi_2(M)$. The symplectic action functional $a_H : \tilde{\mathcal{L}}(M) \rightarrow \mathbb{R}$ is defined by

$$a_H((x, u)) = \int_{D^2} u^* \omega + \int_0^1 H(x(t), t) dt.$$

The Arnold conjecture

Andreas Floer, *Symplectic fixed points and holomorphic spheres*,
Comm. Math. Phys. 120 (1989), no. 4, 575–611.

Theorem. (Floer 1989) Let (P, ω) be a compact symplectic manifold. If I_ω and I_c are proportional, then the fixed point set of every exact diffeomorphism of (P, ω) satisfies the Morse inequalities with respect to any coefficient ring **whenever it is nondegenerate**.

Generalizations: Allowing H to be degenerate (e.g. $H = 0$) leads to critical submanifolds and Morse-Bott homology.

Lagrangian intersection homology

Let $L \subset P$ be a Lagrangian submanifold of a symplectic manifold (P, ω) and $\phi_1 : P \rightarrow P$ a Hamiltonian diffeomorphism such that $\phi_1(L)$ intersects L transversally. There is a Floer chain complex with chain groups generated by the elements of $L \cap \phi_1(L)$ and whose boundary operator is given by counting J -holomorphic strips

$$u : [0, 1] \times \mathbb{R} \rightarrow P, \quad u(\{0\} \times \mathbb{R}) \in L, \quad u(\{1\} \times \mathbb{R}) \in \phi_1(L)$$

Theorem. (Floer 1988) Let P be a compact symplectic manifold, and let $L \subset P$ be a compact Lagrangian submanifold of P with $\pi_2(P, L) = 0$. Moreover, let ϕ_1 be an exact deformation such that $\phi_1(L)$ intersects L transversally. Then $|L \cap \phi_1(L)|$ is bounded below by the sum of the $\mathbb{Z}_2$ -Betti numbers of L .

Lagrangian intersection Floer homology

Extensive work by Fukaya, Oh, Ohta, and Ono involving the Morse-Bott case using [spectral sequences](#), A_∞ algebras, and Kuranishi structures.

Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. **Lagrangian intersection Floer theory: anomaly and obstruction. Part I & II.** AMS/IP Studies in Advanced Mathematics, 46.1 & 46.2. American Mathematical Society, Providence, RI, 2009. xii+396 pp & pp. i–xii and 397–805.

A Morse-Bott version involving [cascades](#) was used by Frauenfelder to prove the Arnold-Givental conjecture for a class of Lagrangians in Marsden-Weinstein quotients.

Urs Frauenfelder. *The Arnold-Givental conjecture and moment Floer homology*. Int. Math. Res. Not. 2004, no. 42, 2179–2269.

Spectral sequence definition

An E^k **spectral sequence** is defined to be a sequence of bigraded modules $\{E_{s,t}^r\}$ over a principal ideal domain R for $r \geq k$, with differentials $d^r : E_{s,t}^r \rightarrow E_{s-r,t+r-1}^r$ that satisfy $(d^r)^2 = 0$. Moreover, if

$$\begin{aligned}\bar{Z}_{s,t}^r &= \ker(d^r : E_{s,t}^r \rightarrow E_{s-r,t+r-1}^r) \\ \bar{B}_{s,t}^r &= \operatorname{im}(d^r : E_{s+r,t-r+1}^r \rightarrow E_{s,t}^r)\end{aligned}$$

then $\bar{B}_{s,t}^r \subseteq \bar{Z}_{s,t}^r$, and there is a given isomorphism

$$E_{s,t}^{r+1} \approx \bar{Z}_{s,t}^r / \bar{B}_{s,t}^r.$$

Note: The differential d^r is only defined on the r^{th} page.

The filtration associated to a Morse-Bott function

For a Morse-Bott function $f : M \rightarrow \mathbb{R}$ and $t \in \mathbb{R}$ we can define the “half-space” $M^t = \{x \in M \mid f(x) \leq t\}$.

If the critical values of f are $c_1 < c_2 < \dots < c_k$, then we have a filtration

$$\emptyset \subseteq M^{c_1} \subseteq M^{c_2} \subseteq \dots \subseteq M^{c_k}.$$

For any $j = 2, \dots, k$, M^{c_j} is homotopic to $M^{c_{j-1}}$ with a λ -disk bundle attached for each critical submanifold of index λ in the critical level $f^{-1}(c_j)$.

This is a generalization of the fact that a Morse function on M determines a CW-complex X that is homotopic to M .

The spectral sequence associated to a filtration

Let (C_*, ∂) a filtered chain complex that is bounded below by $s = 0$. That is, suppose that we have a filtration

$$F_0 C_* \subset \cdots \subset F_{s-1} C_* \subset F_s C_* \subset F_{s+1} C_* \subset \cdots$$

where $F_s C_*$ is a chain subcomplex of C_* for all s . Define

$$\begin{aligned} Z_{s,t}^r &= \{c \in F_s C_{s+t} \mid \partial c \in F_{s-r} C_{s+t-1}\} \\ Z_{s,t}^\infty &= \{c \in F_s C_{s+t} \mid \partial c = 0\}. \end{aligned}$$

The bigraded R -modules in the spectral sequence associated to the filtration are defined to be

$$\begin{aligned} E_{s,t}^r &= Z_{s,t}^r / \left(Z_{s-1,t+1}^{r-1} + \partial Z_{s+r-1,t-r+2}^{r-1} \right) \\ E_{s,t}^\infty &= Z_{s,t}^\infty / \left(Z_{s-1,t+1}^\infty + (\partial C_{s+t+1} \cap F_s C_{s+t}) \right). \end{aligned}$$

The differentials in the spectral sequence

The differential $d^r : E_{s,t}^r \rightarrow E_{s-r,t+r-1}^r$ in the spectral sequence associated to a filtered chain complex is defined by the following diagram.

$$\begin{array}{ccc}
 Z_{s,t}^r & \xrightarrow{\quad \partial \quad} & Z_{s-r,t+r-1}^r \\
 \downarrow & & \downarrow \\
 Z_{s,t}^r / (Z_{s-1,t+1}^{r-1} + \partial Z_{s+r-1,t-r+2}^{r-1}) & \xrightarrow{\quad d^r \quad} & Z_{s-r,t+r-1}^r / (Z_{s-r-1,t+r}^{r-1} + \partial Z_{s-1,t+1}^{r-1}) \\
 \uparrow = & & \uparrow = \\
 E_{s,t}^r & \xrightarrow{\quad d^r \quad} & E_{s-r,t+r-1}^r
 \end{array}$$

The R -module $E_{s,t}^r$ is isomorphic to $\bar{Z}_{s,t}^{r-1} / \bar{B}_{s,t}^{r-1}$ via an isomorphism given by the Noether Isomorphism Theorem.

What the spectral sequence computes

Theorem

If the filtration on the chain complex (C_*, ∂) is convergent and bounded below, then the above spectral sequence converges to the bigraded R -module $GH_*(C_*, \partial)$ associated to the filtration $F_s H_*(C_*, \partial) \equiv \text{im}[H_*(F_s C_*, \partial) \rightarrow H_*(C_*, \partial)]$. That is,

$$E_{s,t}^\infty \approx \bigcap_r Z_{s,t}^r / \bigcup_r \left(Z_{s-1,t+1}^{r-1} + \partial Z_{s+r-1,t-r+2}^{r-1} \right) \approx GH_*(C_*, \partial)_{s,t}$$

where $GH_*(C_*, \partial)_{s,t} \equiv F_s H_{s+t}(C_*, \partial) / F_{s-1} H_{s+t}(C_*, \partial)$.

For a proof, see (for instance) Section 9.1 of Spanier.

Instanton homology and gauge theory

Theorem. (Fukaya 1996) Let N be a connected sum of two homology 3-spheres and $R(N)$ the space of conjugacy classes of $SU(2)$ representations of $\pi_1(N)$. Then $R(N)$ is divided into $R_i(N)$ with $i \in \mathbb{Z}$, and there is a **spectral sequence** with $E_{ij}^1 \cong H_j(R_i; \mathbb{Z})$ such that $E_{ij}^* \implies I_{i+j}(N)$.

Theorem. (Austin-Braam 1996) Suppose N is a 3-manifold such that the Chern-Simons functional may be perturbed to a Morse-Bott function with only reducible critical orbits. With some additional assumptions, certain Donaldson polynomials on a 4-manifold $X = X_1 \cup_N X_2$ vanish.

Frauenfelder 2004 & Bourgeois 2003 (gradient fragments)

Let $f : M \rightarrow \mathbb{R}$ be a Morse-Bott function and suppose

$$\text{Cr}(f) = \coprod_{j=1}^l C_j,$$

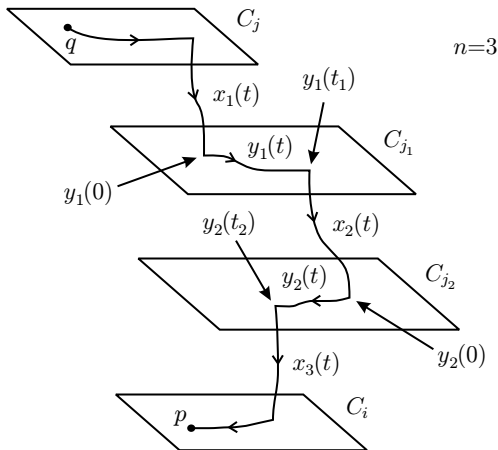
where $C_1, \dots, C_l$ are disjoint connected critical submanifolds of Morse-Bott index $\lambda_1, \dots, \lambda_l$ respectively. Let $f_j : C_j \rightarrow \mathbb{R}$ be a Morse function on the critical submanifold C_j for all $j = 1, \dots, l$.

Definition

If $q \in C_j$ is a critical point of the Morse function $f_j : C_j \rightarrow \mathbb{R}$ for some $j = 1, \dots, l$, then the **total index** of q , denoted λ_q , is defined to be the sum of the Morse-Bott index of C_j and the Morse index of q relative to f_j , i.e.

$$\lambda_q = \lambda_j + \lambda_q^j.$$

A 3-cascade



For $q \in \text{Cr}(f_j)$, $p \in \text{Cr}(f_i)$, and $n \in \mathbb{N}$, a **flow line with n cascades from q to p** is a $2n - 1$ -tuple:

$$((x_k)_{1 \leq k \leq n}, (t_k)_{1 \leq k \leq n-1})$$

where $x_k \in C^\infty(\mathbb{R}, M)$ and $t_k \in \mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$ satisfy the following for all k .

1. Each x_k is a non-constant gradient flow line of f , i.e.

$$\frac{d}{dt}x_k(t) = -(\nabla f)(x_k(t)).$$

2. For the first cascade $x_1(t)$ we have

$$\lim_{t \rightarrow -\infty} x_1(t) \in W_{f_j}^u(q) \subseteq C_j,$$

and for the last cascade $x_n(t)$ we have

$$\lim_{t \rightarrow \infty} x_n(t) \in W_{f_i}^s(p) \subseteq C_i.$$

3. For $1 \leq k \leq n - 1$ there are critical submanifolds C_{j_k} and gradient flow lines $y_k \in C^\infty(\mathbb{R}, C_{j_k})$ of f_{j_k} , i.e.

$$\frac{d}{dt}y_k(t) = -(\nabla f_{j_k})(y_k(t)),$$

such that $\lim_{t \rightarrow \infty} x_k(t) = y_k(0)$ and $\lim_{t \rightarrow -\infty} x_{k+1}(t) = y_k(t_k)$.

Definition

Denote the space of flow lines from q to p with n cascades by $W_n^c(q, p)$, and denote the quotient of $W_n^c(q, p)$ by the action of $\mathbb{R}^n$ by $\mathcal{M}_n^c(q, p) = W_n^c(q, p)/\mathbb{R}^n$. The **set of unparameterized flow lines with cascades from q to p** is defined to be

$$\mathcal{M}^c(q, p) = \bigcup_{n \in \mathbb{Z}_+} \mathcal{M}_n^c(q, p)$$

where $\mathcal{M}_0^c(q, p) = W_0^c(q, p)/\mathbb{R}$.

The $\mathbb{Z}_2$ -cascade chain complex

Define the k^{th} chain group $C_k^c(f)$ to be the free abelian group generated by the critical points of total index k of the Morse-Smale functions f_j for all $j = 1, \dots, l$, and define $n^c(q, p; \mathbb{Z}_2)$ to be the number of flow lines with cascades between a critical point q of total index k and a critical point p of total index $k - 1$ counted mod 2. Let

$$C_*^c(f) \otimes \mathbb{Z}_2 = \bigoplus_{k=0}^m C_k^c(f) \otimes \mathbb{Z}_2$$

and define a homomorphism $\partial_k^c : C_k^c(f) \otimes \mathbb{Z}_2 \rightarrow C_{k-1}^c(f) \otimes \mathbb{Z}_2$ by

$$\partial_k^c(q) = \sum_{p \in Cr(f_{k-1})} n^c(q, p; \mathbb{Z}_2) p.$$

The pair $(C_*^c(f) \otimes \mathbb{Z}_2, \partial_*^c)$ is called the **cascade chain complex** with $\mathbb{Z}_2$ coefficients.

The Arnold-Givental conjecture

Let (M, ω) be a $2n$ -dimensional compact symplectic manifold, $L \subset M$ a compact Lagrangian submanifold, and $R \in \text{Diff}(M)$ an antisymplectic involution, i.e. $R^*\omega = -\omega$ and $R^2 = \text{id}$, whose fixed point set is L .

Conjecture. Let H_t be a smooth family of Hamiltonian functions on M for $0 \leq t \leq 1$ and denote by Φ_H the time-1 map of the flow of the Hamiltonian vector field of H_t . If L intersects $\Phi_H(L)$ transversally, then

$$\#(L \cap \Phi_H(L)) \geq \sum_{k=0}^n b_k(L; \mathbb{Z}_2).$$

Proved by Frauenfelder for a class of Lagrangians in Marsden-Weinstein quotients by letting $H \rightarrow 0$ (2004).

Closed Reeb orbits

Let M be a compact, orientable manifold of dimension $2n - 1$ with contact form α . The **Reeb vector field** R_α associated to the contact form α is characterized by

$$\begin{aligned} d\alpha(R_\alpha, -) &= 0 \\ \alpha(R_\alpha) &= 1. \end{aligned}$$

Closed trajectories of the Reeb vector field are critical points of the action functional $\mathcal{A} : C^\infty(S^1, M) \rightarrow \mathbb{R}$

$$\mathcal{A}(\gamma) = \int_\gamma \alpha.$$

Lemma. For any contact structure ξ on M , there exists a contact form α for ξ such that all closed orbits of R_α are nondegenerate.

Contact homology

Let $\mathbf{A}$ be the graded supercommutative algebra freely generated by the “good” closed Reeb orbits over the graded ring

$\mathbb{Q}[H_2(M; \mathbb{Z})/\mathcal{R}]$, i.e. $\gamma_1 \gamma_2 = (-1)^{|\gamma_1||\gamma_2|} \gamma_2 \gamma_1$.

Theorem. (Eliashberg-Hofer 2000) There is a differential $d : \mathbf{A} \rightarrow \mathbf{A}$ defined by counting J -holomorphic curves in the symplectization $(\mathbb{R} \times M, d(e^t \alpha))$ such that $(\mathbf{A}, d)$ is a **differential graded algebra**. Moreover, $HC_*(M, \xi) \stackrel{\text{def}}{=} H_*(\mathbf{A}, d)$ is an invariant of the contact structure ξ .

Theorem. (Bourgeois 2003) Assume that α is a contact form of **Morse-Bott type** for (M, ξ) and that J is an almost complex structure on the symplectization that is S^1 -invariant along the critical submanifolds N_T . Then there is a chain complex with a boundary operator defined by counting **cascades** whose homology is isomorphic to the contact homology $HC_*(M, \xi)$.

The Yang-Mills gradient flow

Let (Σ, g) be a closed oriented Riemann surface, G a compact Lie group, $\mathfrak{g}$ its Lie algebra, and P a principal G -bundle over Σ . Pick an ad-invariant inner product on $\mathfrak{g}$, let $\mathcal{A}(P)$ denote the affine space of $\mathfrak{g}$ -valued connection 1-forms on P , and define $\mathcal{YM} : \mathcal{A}(P) \rightarrow \mathbb{R}$ by

$$\mathcal{YM}(A) = \int_{\Sigma} F_A \wedge *F_A$$

where $F_A = dA + \frac{1}{2}[A \wedge A]$ is the curvature of A .

The Yang-Mills function is a Morse-Bott function studied by Atiyah-Bott and by Swoboda (2012) using cascades.

The explicit perturbation of $f : M \rightarrow \mathbb{R}$

Let T_j be a small tubular neighborhood around each connected component $C_j \subseteq Cr(f)$ for all $j = 1, \dots, l$. Pick a positive Morse function $f_j : C_j \rightarrow \mathbb{R}$ and extend f_j to a function on T_j by making f_j constant in the direction normal to C_j for all $j = 1, \dots, l$.

Let $\tilde{T}_j \subset T_j$ be a smaller tubular neighborhood of C_j with the same coordinates as T_j , and let ρ_j be a smooth bump function which is constant in the coordinates parallel to C_j , equal to 1 on $\tilde{T}_j$, equal to 0 outside of T_j , and decreases on $T_j - \tilde{T}_j$ as the coordinates move away from C_j . For small $\varepsilon > 0$ (and a careful choice of the metric) this determines a Morse-Smale function

$$h_\varepsilon = f + \varepsilon \left(\sum_{j=1}^l \rho_j f_j \right).$$

Identical chain groups

For every sufficiently small $\varepsilon > 0$ and $k = 0, \dots, m$ we have

$$C_k^c(f) \approx C_k(h_\varepsilon) = \bigoplus_{\lambda_j + n = k} C_n(f_j).$$

Is $\mathcal{M}^c(q, p) \approx \mathcal{M}_{h_\varepsilon}(q, p)$ when $\lambda_q - \lambda_p = 1$?

If so, then we can use the orientations on $\mathcal{M}_{h_\varepsilon}(q, p)$ to define the cascade chain complex over $\mathbb{Z}$ so that $\partial_k^c = -\partial_k$ for all $k = 0, \dots, m$, where ∂_k is the Morse-Smale-Witten boundary operator of h_ε . In particular,

$$H_*((C_*(f), \partial_*^c)) \approx H_*(M; \mathbb{Z}).$$

Theorem (Banyaga-H 2011)

Assume that f satisfies the Morse-Bott-Smale transversality condition with respect to the Riemannian metric g on M , $f_k : C_k \rightarrow \mathbb{R}$ satisfies the Morse-Smale transversality condition with respect to the restriction of g to C_k for all $k = 1, \dots, l$, and the unstable and stable manifolds $W_{f_j}^u(q)$ and $W_{f_i}^s(p)$ are transverse to the beginning and endpoint maps.

1. When $n = 0, 1$ the set $\mathcal{M}_n^c(q, p)$ is either empty or a smooth manifold without boundary.
2. For $n > 1$ the set $\mathcal{M}_n^c(q, p)$ is either empty or a smooth manifold with corners.
3. The set $\mathcal{M}^c(q, p)$ is either empty or a smooth manifold without boundary.

In each case the dimension of the manifold is $\lambda_q - \lambda_p - 1$. The above manifolds are orientable when M and C_k are orientable.

Compactness

Denote the space of nonempty closed subsets of $M \times \overline{\mathbb{R}}^l$ in the topology determined by the Hausdorff metric by $\mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$, and map a broken flow line with cascades $(v_1, \dots, v_n)$ to its image $\text{Im}(v_1, \dots, v_n) \subset M$ and the time t_j spent flowing along or resting on each critical submanifold C_j for all $j = 1, \dots, l$.

Theorem (Banyaga-H 2011)

The space $\overline{\mathcal{M}}^c(q, p)$ of broken flow lines with cascades from q to p is compact, and there is a continuous embedding

$$\mathcal{M}^c(q, p) \hookrightarrow \overline{\mathcal{M}}^c(q, p) \subset \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l.$$

Hence, every sequence of unparameterized flow lines with cascades from q to p has a subsequence that converges to a broken flow line with cascades from q to p .

Correspondence of moduli spaces

Theorem (Banyaga-H 2011)

Let $p, q \in \text{Cr}(h_\varepsilon)$ with $\lambda_q - \lambda_p = 1$. For any sufficiently small $\varepsilon > 0$ there is a bijection between unparameterized cascades and unparameterized gradient flow lines of the Morse-Smale function $h_\varepsilon : M \rightarrow \mathbb{R}$ between q and p ,

$$\mathcal{M}^c(q, p) \leftrightarrow \mathcal{M}_{h_\varepsilon}(q, p).$$

Definition

Let $p, q \in \text{Cr}(h_\varepsilon)$ with $\lambda_q - \lambda_p = 1$, define an orientation on the zero dimensional manifold $\mathcal{M}^c(q, p)$ by identifying it with the left hand boundary of $\mathcal{M}_{h_\varepsilon}(q, p) \times [0, \varepsilon]$.

The Morse-Bott chain complex with cascades

Define the k^{th} chain group $C_k^c(f)$ to be the free abelian group generated by the critical points of total index k of the Morse-Smale functions f_j for all $j = 1, \dots, l$, and define $n^c(q, p)$ to be the number of flow lines with cascades between a critical point q of total index k and a critical point p of total index $k - 1$ counted with signs determined by the orientations. Let

$$C_*^c(f) = \bigoplus_{k=0}^m C_k^c(f)$$

and define a homomorphism $\partial_k^c : C_k^c(f) \rightarrow C_{k-1}^c(f)$ by

$$\partial_k^c(q) = \sum_{p \in Cr(f_{k-1})} n^c(q, p)p.$$

Correspondence of chain complexes

Theorem (Banyaga-H 2011)

For $\varepsilon > 0$ sufficiently small we have $C_k^c(f) = C_k(h_\varepsilon)$ and $\partial_k^c = -\partial_k$ for all $k = 0, \dots, m$, where ∂_k denotes the Morse-Smale-Witten boundary operator determined by the Morse-Smale function h_ε . In particular, $(C_*^c(f), \partial_*^c)$ is a chain complex whose homology is isomorphic to the singular homology $H_*(M; \mathbb{Z})$.

Moral: The cascade chain complex of a Morse-Bott-Smale function $f : M \rightarrow \mathbb{R}$ **is the same** as the Morse-Smale-Witten complex of a small perturbation of f .

Note: A similar result was proved in the context of symplectic homology by Bourgeois and Oancea (2009).

Multicomplexes

Let R be a principal ideal domain. A first quadrant **multicomplex** X is a bigraded R -module $\{X_{p,q}\}_{p,q \in \mathbb{Z}_+}$ with differentials

$$d_i : X_{p,q} \rightarrow X_{p-i,q+i-1} \quad \text{for all } i = 0, 1, \dots$$

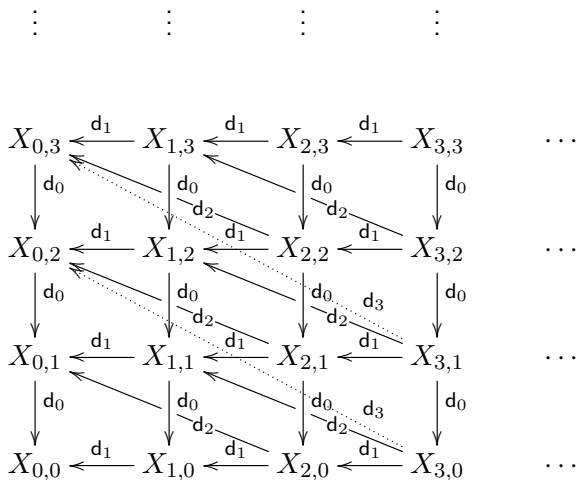
that satisfy

$$\sum_{i+j=n} d_i d_j = 0 \quad \text{for all } n.$$

A first quadrant multicomplex can be **assembled** to form a filtered chain complex $((CX)_*, \partial)$ by summing along the diagonals, i.e.

$$(CX)_n \equiv \bigoplus_{p+q=n} X_{p,q} \quad \text{and} \quad F_s(CX)_n \equiv \bigoplus_{\substack{p+q=n \\ p \leq s}} X_{p,q}$$

and $\partial_n = d_0 \oplus \dots \oplus d_n$ for all $n \in \mathbb{Z}_+$. The above relations then imply that $\partial_n \circ \partial_{n+1} = 0$ and $\partial_n(F_s(CX)_*) \subseteq F_s(CX)_*$.



A bicomplex has two filtrations, but a general multicomplex only has one filtration.

$$\begin{array}{ccccccc}
 \dots & X_{3,0} & \xrightarrow{d_0} & 0 & & & \\
 & \oplus & & \oplus & & & \\
 \dots & X_{2,1} & \xrightarrow{d_0} & X_{2,0} & \xrightarrow{d_0} & 0 & \\
 & \oplus & & \oplus & & \oplus & \\
 \dots & X_{1,2} & \xrightarrow{d_0} & X_{1,1} & \xrightarrow{d_0} & X_{1,0} & \xrightarrow{d_0} 0 \\
 & \oplus & & \oplus & & \oplus & \\
 \dots & X_{0,3} & \xrightarrow{d_0} & X_{0,2} & \xrightarrow{d_0} & X_{0,1} & \xrightarrow{d_0} X_{0,0} \xrightarrow{d_0} 0 \\
 & \parallel & & \parallel & & \parallel & \\
 \dots & (CX)_3 & \xrightarrow{\partial_3} & (CX)_2 & \xrightarrow{\partial_2} & (CX)_1 & \xrightarrow{\partial_1} (CX)_0 \xrightarrow{\partial_0} 0
 \end{array}$$

Diagram illustrating a multicomplex structure with differentials d_i and a spectral sequence structure with differentials ∂_i . The diagram shows a grid of objects $X_{i,j}$ and $(CX)_i$ connected by differentials. A dotted line indicates a specific path or relationship between the objects.

The bigraded module associated to the filtration

$$F_s(CX)_n \equiv \bigoplus_{\substack{p+q=n \\ p \leq s}} X_{p,q}$$

is

$$G((CX)_*)_{s,t} = F_s(CX)_{s+t} / F_{s-1}(CX)_{s+t} \approx X_{s,t}$$

for all $s, t \in \mathbb{Z}_+$, and the E^1 term of the associated spectral sequence is given by

$$E_{s,t}^1 = Z_{s,t}^1 / (Z_{s-1,t+1}^0 + \partial Z_{s,t+1}^0)$$

where

$$\begin{aligned} Z_{s,t}^1 &= \{c \in F_s(CX)_{s+t} \mid \partial c \in F_{s-1}(CX)_{s+t-1}\} \\ Z_{s,t}^0 &= \{c \in F_s(CX)_{s+t} \mid \partial c \in F_s(CX)_{s+t-1}\} = F_s(CX)_{s+t}. \end{aligned}$$

$E_{s,t}^1$ and d^1 are induced from d_0 and d_1

Theorem

Let $(\{X_{p,q}\}_{p,q \in \mathbb{Z}_+}, \{d_i\}_{i \in \mathbb{Z}_+})$ be a first quadrant multicomplex and $((CX)_*, \partial)$ the associated assembled chain complex. Then the E^1 term of the spectral sequence associated to the filtration of $(CX)_*$ determined by the restriction $p \leq s$ is given by $E_{s,t}^1 \approx H_{s+t}(X_{s,*}, d_0)$ where $(X_{s,*}, d_0)$ denotes the following chain complex.

$$\cdots \xrightarrow{d_0} X_{s,3} \xrightarrow{d_0} X_{s,2} \xrightarrow{d_0} X_{s,1} \xrightarrow{d_0} X_{s,0} \xrightarrow{d_0} 0$$

Moreover, the d^1 differential on the E^1 term of the spectral sequence is induced from the homomorphism d_1 in the multicomplex.

However, d^r is not induced from d_r for $r \geq 2$

Consider the following first quadrant double complex

$$\begin{array}{ccccc}
 0 & \xleftarrow{d_1} & 0 & \xleftarrow{d_1} & 0 \\
 \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\
 \langle x_{0,1} \rangle & \xleftarrow{d_1} & \langle x_{1,1} \rangle & \xleftarrow{d_1} & 0 \\
 \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\
 0 & \xleftarrow{d_1} & \langle x_{1,0} \rangle & \xleftarrow{d_1} & \langle x_{2,0} \rangle
 \end{array}$$

where $\langle x_{p,q} \rangle$ denotes the free abelian group generated by $x_{p,q}$, the groups $X_{p,q} = 0$ for $p + q > 2$, and the homomorphisms d_0 and d_1 satisfy the following: $d_0(x_{1,1}) = x_{1,0}$, $d_1(x_{1,1}) = x_{0,1}$, and $d_1(x_{2,0}) = x_{1,0}$. In this case, $d_2 = 0$ but $d^2 \neq 0$.

Morse-Bott-Smale functions

Definition (Morse-Bott-Smale Transversality)

A function $f : M \rightarrow \mathbb{R}$ is said to satisfy the **Morse-Bott-Smale transversality** condition with respect to a Riemannian metric g on M if and only if f is Morse-Bott and for any two connected critical submanifolds B and B' , $W^u(p)$ intersects $W^s(B')$ transversely for all $p \in B$, i.e. $W^u(p) \pitchfork W^s(B')$.

[Zhou (2024)] Given a Morse-Bott function $f : M \rightarrow \mathbb{R}$ it may not be possible to pick a Riemannian metric for which f is M-B-S. However, it is always possible to find a **Morse-Bott-Smale pair** (f, g) , where $W^u(B_i) \pitchfork W^s(B_j)$ for all i, j and the maps

$$W(B_{i_1}, B_{i_2}) \times_{B_{i_2}} \cdots \times_{B_{i_{l-1}}} W(B_{i_{l-1}}, B_{i_l}) \xrightarrow{\partial_+} B_{i_l} \xleftarrow{\partial_-} W(B_{i_l}, B_{i_{l+1}})$$

(and their iterates) are transverse for all $(i_1, \dots, i_{l+1})$.

Algebraic structure of a Morse-Bott-Smale chain complex

Assume that $f : M \rightarrow \mathbb{R}$ is a Morse-Bott-Smale function and the manifold M , the critical submanifolds, and their negative normal bundles are all orientable.

Let $C_p(B_i)$ be the group of “ p -dimensional chains” in the critical submanifolds of index i . For all $k = 0, \dots, m$ define the group of chains of Morse-Bott degree k to be

$$C_k(f) = \bigoplus_{i=0}^m C_{k-i}(B_i).$$

The boundary operator is defined as a sum of homomorphisms $\partial = \partial_0 \oplus \dots \oplus \partial_m$ where $\partial_j : C_p(B_i) \rightarrow C_{p+j-1}(B_{i-j})$.

The Morse-Bott-Smale multicomplex

$$\begin{array}{ccccccc}
 \cdots & C_0(B_3) & \xrightarrow{\partial_0} & 0 & & & \\
 & \oplus & \searrow^{\partial_3} & \searrow^{\partial_1} & \oplus & & \\
 \cdots & C_1(B_2) & \xrightarrow{\partial_0} & C_0(B_2) & \xrightarrow{\partial_0} & 0 & \\
 & \oplus & \searrow^{\partial_2} & \searrow^{\partial_1} & \oplus & \searrow^{\partial_1} & \oplus \\
 \cdots & C_2(B_1) & \xrightarrow{\partial_0} & C_1(B_1) & \xrightarrow{\partial_0} & C_0(B_1) & \xrightarrow{\partial_0} & 0 \\
 & \oplus & \searrow^{\partial_2} & \searrow^{\partial_1} & \oplus & \searrow^{\partial_2} & \searrow^{\partial_1} & \oplus \\
 \cdots & C_3(B_0) & \xrightarrow{\partial_0} & C_2(B_0) & \xrightarrow{\partial_0} & C_1(B_0) & \xrightarrow{\partial_0} & C_0(B_0) & \xrightarrow{\partial_0} & 0 \\
 & \parallel & & \parallel & & \parallel & & \parallel & \\
 \cdots & C_3(f) & \xrightarrow{\partial} & C_2(f) & \xrightarrow{\partial} & C_1(f) & \xrightarrow{\partial} & C_0(f) & \xrightarrow{\partial} & 0
 \end{array}$$

The Austin-Braam cochain complex (1995)

Let B_i be the set of critical points of index i and $C^{i,j} = \Omega^j(B_i)$ the set of j -forms on B_i . Austin and Braam define maps

$$\partial_r : C^{i,j} \rightarrow C^{i+r,j-r+1}$$

for $r = 0, 1, 2, \dots, m$ which raise the “total degree” $i + j$ by one.

The maps $\partial_r : \Omega^j(B_i) \rightarrow \Omega^{j-r+1}(B_{i+r})$ fit together to form a cochain complex where $\partial = \partial_0 \oplus \dots \oplus \partial_m$ and

$$C^k(f) = \bigoplus_{i=0}^k \Omega^{k-i}(B_i).$$

The Austin-Braam cochain multicomplex

$$\begin{array}{ccccccc}
 & & & & \Omega^0(B_3) & & \\
 & & & \nearrow \partial_1 & & \nearrow \partial_1 & \\
 & & \Omega^0(B_2) & \xrightarrow{d} & \Omega^1(B_2) & & \\
 & \nearrow \partial_1 & & \nearrow \partial_1 & & \nearrow \partial_1 & \\
 & \Omega^0(B_1) & \xrightarrow{d} & \Omega^1(B_1) & \xrightarrow{d} & \Omega^2(B_1) & \\
 & \nearrow \partial_1 & \nearrow \partial_2 & \nearrow \partial_1 & \nearrow \partial_1 & \nearrow \partial_1 & \\
 \Omega^0(B_0) & \xrightarrow{d} & \Omega^1(B_0) & \xrightarrow{d} & \Omega^2(B_0) & \xrightarrow{d} & \Omega^3(B_0) \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 C^0(f) & \xrightarrow{\partial} & C^1(f) & \xrightarrow{\partial} & C^2(f) & \xrightarrow{\partial} & C^3(f) \xrightarrow{\partial} \dots
 \end{array}$$

The Austin-Braam coboundary operator

Pulling back along the endpoint map and then apply integration along the fiber using the beginning point map gives a “pull-push” operation that transports a differential form from B_i to B_{i+r} .

$$B_{i+r} \xleftarrow{\partial_-} \overline{\mathcal{M}}(B_{i+r}, B_i) \xrightarrow{\partial_+} B_i$$

Definition (Austin-Braam)

The map $\partial_r : \Omega^j(B_i) \rightarrow \Omega^{j-r+1}(B_{i+r})$ is defined by

$$\partial_r(\omega) = \begin{cases} d\omega & r = 0 \\ (-1)^j(\partial_-)_*(\partial_+^*\omega) & r \neq 0. \end{cases}$$

Note: Integration along the fiber requires the “fibration condition” [Zhou (2024)], which is a consequence of M-B-S transversality.

The Banyaga-H approach (2010)

Modeled on **cubical singular homology**. Based on ideas from Austin and Braam (~ 1995), Barraud and Cornea (~ 2004), Fukaya (~ 1995), Weber (~ 2006) etc.

Step 1: Generalize the notion of singular p -simplexes to allow maps from spaces other than the standard p -simplex $\Delta^p \subset \mathbb{R}^{p+1}$ or the unit p -cube $I^p \subset \mathbb{R}^p$. These generalizations of Δ^p (or I^p) are called **abstract topological chains**, and the corresponding singular chains are called **singular topological chains**.

Step 2: Show that ∂ extends to fibered products, and show that the compactified moduli spaces of gradient flow lines are abstract topological chains, i.e. ∂_0 is defined.

The Banyaga-H approach (2010)

Step 3: Define the set of **allowed domains** C_p in the Morse-Bott-Smale chain complex as a collection of fibered products (with ∂_0 defined), and show that the allowed domains are all compact oriented smooth manifolds with corners.

Step 4: Define $\partial_1, \dots, \partial_m$ using fibered products of compactified moduli spaces of gradient flow lines and the beginning and endpoint maps.

If $\sigma : P \rightarrow B_i$ is a singular C_p -space in $S_p^\infty(B_i)$, then for any $j = 1, \dots, i$ composing the projection map π_2 onto the second component of $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ with the endpoint map $\partial_+ : \overline{\mathcal{M}}(B_i, B_{i-j}) \rightarrow B_{i-j}$ gives a map

$$P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\pi_2} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\partial_+} B_{i-j}.$$

The Banyaga-H approach (2010)

Step 5: Define **orientation conventions** on the elements of C_p and corresponding **degeneracy relations** to identify singular topological chains that are “essentially” the same. Show that $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ is compatible with the degeneracy relations.

Step 6: Show that the homology of the Morse-Bott-Smale chain complex $(C_*(f), \partial_*)$ is independent of $f : M \rightarrow \mathbb{R}$.

When $f : M \rightarrow \mathbb{R}$ is Morse-Smale, $(C_*(f), \partial_*)$ is the Morse-Smale-Witten complex, and when f is constant $(C_*(f), \partial_*)$ is the chain complex of cubical singular chains. This gives a new proof of the Morse Homology Theorem.

$$\begin{array}{ccccccc}
 \text{---} & C_0(B_3) & \xrightarrow{\partial_0} & 0 \\
 & \oplus \nearrow \partial_3 & & \oplus \\
 & \partial_2 \searrow & \partial_1 \nearrow & \\
 \text{---} & C_1(B_2) & \xrightarrow{\partial_0} & C_0(B_2) & \xrightarrow{\partial_0} & 0 \\
 & \oplus \nearrow \partial_2 & & \oplus & \nearrow \partial_1 & \oplus \\
 & \partial_1 \searrow & \partial_0 \nearrow & & \partial_2 \searrow & \\
 \text{---} & C_2(B_1) & \xrightarrow{\partial_0} & C_1(B_1) & \xrightarrow{\partial_0} & C_0(B_1) & \xrightarrow{\partial_0} & 0 \\
 & \oplus \nearrow \partial_2 & & \oplus & \nearrow \partial_1 & \oplus & \nearrow \partial_1 & \oplus \\
 & \partial_1 \searrow & \partial_0 \nearrow & & \partial_1 \searrow & & \partial_1 \searrow & \\
 \text{---} & C_3(B_0) & \xrightarrow{\partial_0} & C_2(B_0) & \xrightarrow{\partial_0} & C_1(B_0) & \xrightarrow{\partial_0} & C_0(B_0) & \xrightarrow{\partial_0} & 0 \\
 & \parallel & & \parallel & & \parallel & & \parallel & & \\
 \text{---} & C_3(f) & \xrightarrow{\partial} & C_2(f) & \xrightarrow{\partial} & C_1(f) & \xrightarrow{\partial} & C_0(f) & \xrightarrow{\partial} & 0
 \end{array}$$

Recent developments

Zhengyi Zhou, *Morse-Bott cohomology from homological perturbation theory*, *Algebr. Geom. Topol.* **24** (2024), no. 3, 1321–1429.

“...we explain how homological perturbation theory is used in Morse–Bott theories, and both our construction and the cascades construction can be interpreted as applications of homological perturbations.”

Uses currents and coefficients in $\mathbb{R}$. Zhou explains “...how to heuristically interpret the cascades cochain complex as a homological perturbation on the Austin–Braam cochain complex.”

References

- ▶ David Austin and Peter Braam, *Morse-Bott theory and equivariant cohomology*, **The Floer memorial volume**, Progr. Math. **133** (1995), 123–183.
- ▶ David Austin and Peter Braam, *Equivariant Floer theory and gluing Donaldson polynomials*, Topology **35** (1996), no. 1, 167–200.
- ▶ Augustin Banyaga and David Hurtubise, *A proof of the Morse-Bott Lemma*, Expo. Math. **22**, no. 4, 365–373, 2004.
- ▶ Augustin Banyaga and David Hurtubise, **Lectures on Morse homology**, Kluwer Texts in the Mathematical Sciences **29**, 2004.
- ▶ Augustin Banyaga and David Hurtubise, *The Morse-Bott inequalities via a dynamical systems approach*, Ergodic Theory Dynam. Systems **29** (2009), no. 6, 1693–1703.

References

- ▶ Augustin Banyaga and David Hurtubise, *Morse-Bott homology*, Trans. Amer. Math. Soc., **362** (2010), no. 8, 3997–4043.
- ▶ Augustin Banyaga and David Hurtubise, *Cascades and perturbed Morse-Bott functions*, preprint, 2011.
- ▶ Frédéric Bourgeois and Alexandru Oancea, *Symplectic homology, autonomous Hamiltonians, and Morse-Bott moduli spaces*, Duke Math. J. **146** (2009), no. 1, 74–174.
- ▶ Frédéric Bourgeois and Alexandru Oancea, *An exact sequence for contact and symplectic homology*, Invent. Math. **179** (2009), no. 3, 611–680.
- ▶ Andreas Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988), no. 2, 215–240.

References

- ▶ Andreas Floer, *Instanton homology, surgery, and knots*, **Geometry of low-dimensional manifolds**, London Math. Soc. Lecture Note Ser. **150** (1990), 97–114.
- ▶ Andreas Floer, *Morse theory for Lagrangian intersections*, Journal of Differential Geom **28** (1988), 513–547.
- ▶ Urs Frauenfelder, *The Arnold-Givental conjecture and moment Floer homology*, Int. Math. Res. Not. **2004**, no. 42, 2179–2269.
- ▶ Kenji Fukaya, *Floer homology of connected sum of homology 3-spheres*, Topology **35** (1996), no. 1, 89–136.
- ▶ David Hurtubise, *Multicomplexes and spectral sequences*, J. Algebra Appl. **9** (2010), no. 4, 519–530.

References

- ▶ Muriel Livernet, Sarah Whitehouse, and Stephanie Ziegenhagen, *On the spectral sequence associated to a multicomplex*, J. Pure Appl. Algebra **224** (2020), no. 2, 528–535.
- ▶ Jean-Pierre Meyer, *Acyclic models for multicomplexes*, Duke Math. J. **45** (1978), no. 1, 67–85.
- ▶ Ryuma Orita and Kanon Yashiro, *Morse-Bott-Smale chain complex*, arXiv:2504.16962v2 [math.AT] (2025).
- ▶ Yongbin Ruan and Gang Tian, *Bott-type symplectic Floer cohomology and its multiplication structures*, Math. Res. Lett. **2** (1995), no. 2, 203–219.
- ▶ Jan Swoboda, *Morse homology for the Yang-Mills gradient flow*, J. Math. Pures Appl. (9) **98** (2012), no. 2, 160–210.

References

- ▶ Gang Liu and Gang Tian, *On the equivalence of multiplicative structures in Floer homology and quantum homology*, Acta Math. Sin. **15** (1999), no. 1, 53–80.
- ▶ Joa Weber, *The Morse-Witten complex via dynamical systems*, Expo. Math. **24** (2006), no. 2, 127–159.
- ▶ Edward Witten, **Supersymmetry and Morse theory**, J. Differential Geom. **17** (1982), no. 4, 661–692 (1983).
- ▶ Zhengyi Zhou, *Morse-Bott cohomology from homological perturbation theory*, Algebr. Geom. Topol. **24** (2024), no. 3, 1321–1429.