"Singular" Morse-Bott Homology

Augustin Banyaga and David Hurtubise

Penn State Altoona
altoona.psu.edu/math

May 4, 2023

The project

Construct a "singular" chain complex analogous to the Morse-Smale-Witten chain complex for Morse-Bott functions.

Question: Why would anyone want to do this?

After all, we can always perturb a smooth function to get a Morse-Smale function. Also, a Morse-Bott function determines a filtration, and hence, a spectral sequence.

Perturbations

1. If $f : M \to \mathbb{R}$ is a Morse-Bott function, study the Morse-Smale-Witten complex as $\varepsilon \to 0$ of

$$h = f + \varepsilon \left(\sum_{j=1}^{l} \rho_j f_j \right).$$

2. If $h : M \to \mathbb{R}$ is a Morse-Smale function, study the Morse-Smale-Witten complex of $\varepsilon h : M \to \mathbb{R}$ as $\varepsilon \to 0$.

Equivariant homology

If $\pi : E \to B$ is a smooth fiber bundle with fiber F and f is a Morse function on B, then $f \circ \pi$ is a Morse-Bott function with critical submanifolds diffeomorphic to F.

$$\begin{array}{ccccc} F & \longrightarrow & E & & \\ & & \Big\downarrow \pi & & \\ & & B & \xrightarrow{\ f\ } & \mathbb{R} \end{array}$$

In particular, if G is a Lie group acting on M and $\pi : EG \to BG$ is the classifying bundle for G, then this might be useful for studying equivariant homology $H_*^G(M) := H_*(EG \times_G M)$.

$$\begin{array}{ccccc} M & \longrightarrow & EG \times_G M & & \\ & & \Big\downarrow \pi & & \\ & & BG & \xrightarrow{\ f\ } & \mathbb{R} \end{array}$$

Morse-Bott functions

Definition

A smooth function $f : M \to \mathbb{R}$ on a smooth manifold M is called a **Morse-Bott function** if and only if $\mathrm{Cr}(f)$ is a disjoint union of connected submanifolds, and for each connected submanifold $B \subseteq \mathrm{Cr}(f)$ the normal Hessian is non-degenerate for all $p \in B$.

Lemma (Morse-Bott Lemma)

Let $f : M \to \mathbb{R}$ be a Morse-Bott function, and let B be a critical submanifold. For any $p \in B$ there is a local chart of M around p and a local splitting of the normal bundle $\nu_*(B) = \nu_*^+(B) \oplus \nu_*^-(B)$ identifying a point $x \in M$ in its domain with $(u, v, w) \in B \oplus \nu_*^+(B) \oplus \nu_*^-(B)$ such that within this chart f assumes the form

$$f(x) = f(u, v, w) = f(B) + |v|^2 - |w|^2.$$

Stable/Unstable manifolds

Pick a Riemannian metric g on M, and let φ_t be the flow of $-\nabla f$. For $p \in Cr(f)$ the **stable manifold** $W^s(p)$ and the **unstable manifold** $W^u(p)$ are defined as follows.

$$W^s(p) = \{x \in M | \lim_{t\to\infty} \varphi_t(x) = p\} \qquad (f \text{ decreases to } p)$$
$$W^u(p) = \{x \in M | \lim_{t\to-\infty} \varphi_t(x) = p\} \qquad (f \text{ increases to } p)$$

Definition

If $f : M \to \mathbb{R}$ is a Morse-Bott function, then the stable and unstable manifolds of a critical submanifold B are defined to be

$$W^s(B) = \bigcup_{p\in B} W^s(p)$$
$$W^u(B) = \bigcup_{p\in B} W^u(p).$$

Theorem (Stable/Unstable Manifold Theorem)

The stable and unstable manifolds $W^s(B)$ and $W^u(B)$ are the surjective images of smooth injective immersions $E^+ : \nu_*^+(B) \to M$ and $E^- : \nu_*^-(B) \to M$. There are smooth endpoint maps $\partial_+ : W^s(B) \to B$ and $\partial_- : W^u(B) \to B$ given by $\partial_+(x) = \lim_{t\to\infty} \varphi_t(x)$ and $\partial_-(x) = \lim_{t\to-\infty} \varphi_t(x)$ which when restricted to a neighborhood of B have the structure of locally trivial fiber bundles.

Definition

The **index** of B is the dimension of $\nu_*^-(B)$.

Morse-Bott-Smale functions

Definition (Morse-Bott-Smale Transversality)

A function $f : M \to \mathbb{R}$ is said to satisfy the **Morse-Bott-Smale transversality** condition with respect to a Riemannian metric g on M if and only if f is Morse-Bott and for any two connected critical submanifolds B and B', $W^u(p)$ intersects $W^s(B')$ transversely for all $p \in B$, i.e. $W^u(p) \pitchfork W^s(B')$.

[Zhou] Given a Morse-Bott function $f : M \to \mathbb{R}$ it may not be possible to pick a Riemannian metric for which f is M-B-S. However, it is always possible to find a *Morse-Bott-Smale pair* (f, g), where $W^u(B_i) \pitchfork W^s(B_j)$ for all i,j and the maps

$$W(B_{i_1}, B_{i_2}) \times_{B_{i_2}} \cdots \times_{B_{i_{l-1}}} W(B_{i_{l-1}}, B_{i_l}) \overset{\partial_+}{\to} B_{i_l} \overset{\partial_-}{\leftarrow} W(B_{i_l}, B_{i_{l+1}})$$

are transverse for all $(i_1, \ldots, i_{l+1})$.

Counterexample [Latschev]

Consider the function $h : T^2 \to [-6, 0]$ given by

$$h(\varphi, \psi) = -(2 + \cos 2\varphi)(1 + \cos\psi)$$

for $(\varphi, \psi) \in [0, 2\pi) \times [0, 2\pi)$.

$h_\varphi = (2 \sin 2\varphi)(1 + \cos\psi)$ and $h_\psi = (2 + \cos 2\varphi)(\sin\psi)$

The maximum value 0 determines a critical submanifold of dimension one: $B_1 = \{(\varphi, \pi)|\ \varphi \in [0, 2\pi)\}$, and there are two discrete minima $(0, 0)$ and $(\pi, 0)$ and two discrete saddle points $(\pi/2, 0)$ and $(3\pi/2, 0)$. There is no metric such that $W^u(p) \pitchfork W^s((\pi/2, 0))$ for $p \in B_1$ because both $W^u(p)$ and $W^s((\pi/2, 0))$ are one dimensional. However, it is possible to pick a metric such that $-h$ is Morse-Bott-Smale.

Dimensions, index, and coindex

Lemma

If f satisfies the Morse-Bott-Smale transversality condition, B is a critical submanifold of dimension b, the index of B is $\lambda_B = \text{dim } \nu_*^-(B)$, and the coindex of B is $\lambda_B^* = \text{dim } \nu_*^+(B)$, then

$$\begin{aligned} m &= b + \lambda_B^* + \lambda_B \\ \text{dim } W^u(B) &= b + \lambda_B \\ \text{dim } W^s(B') &= b' + \lambda_{B'}^* = m - \lambda_{B'} \\ \text{dim } W(B,B') &= \lambda_B - \lambda_{B'} + b \quad (\text{if } W(B,B') \neq \emptyset), \end{aligned}$$

where $m = \text{dim } M$.

Note: The dimension of $W(B, B')$ does not depend on the dimension of the critical submanifold B'.

Algebraic structure of a Morse-Bott-Smale chain complex

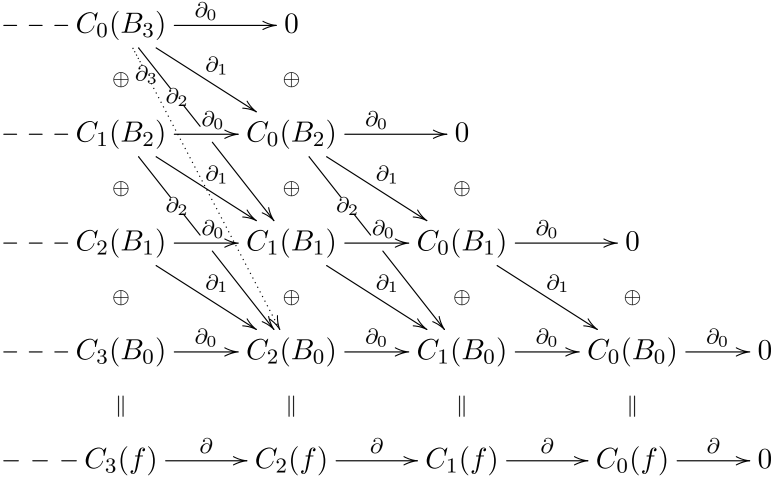
Assume that $f : M \to \mathbb{R}$ is a Morse-Bott-Smale function and the manifold M, the critical submanifolds, and their negative normal bundles are all orientable.

Let $C_p(B_i)$ be the group of "p-dimensional chains" in the critical submanifolds of index i. For all $k = 0, \ldots, m$ define the group of chains of Morse-Bott degree k to be

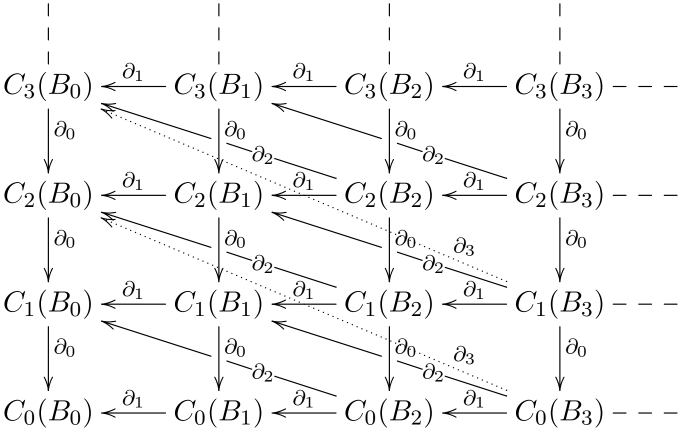
$$C_k(f) = \bigoplus_{i=0}^{m} C_{k-i}(B_i).$$

The boundary operator is defined as a sum of homomorphisms $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ where $\partial_j : C_p(B_i) \to C_{p+j-1}(B_{i-j})$.

The Morse-Bott-Smale chain complex



The Morse-Bott-Smale multicomplex



The spectral sequence associated to a M-B-S complex

The Morse-Bott chain complex $(C_*(f), \partial)$ is a filtered differential graded $\mathbb{Z}$-module where the (increasing) filtration is determined by the Morse-Bott index.

$$F_s C_k(f) \equiv \bigoplus_{i \leq s} C_{k-i}(B_i)$$

The associated bigraded module $G(C_*(f))$ is given by

$$G(C_*(f))_{s,t} = F_s C_{s+t}(f)/F_{s-1}C_{s+t}(f) \approx C_t(B_s),$$

and E^1 term of the associated spectral sequence is

$$E^1_{s,t} \approx H_{s+t}(F_s C_*(f)/F_{s-1}C_*(f)) \approx H_t(B_s),$$

where the homology is computed with respect to the boundary operator on the chain complex $F_s C_*(f)/F_{s-1}C_*(f)$ induced by $\partial = \partial_0 \oplus \cdots \oplus \partial_m$, i.e. ∂_0.

The E^1 term of the spectral sequence

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
H_2(B_0) & \xleftarrow{d_1} & H_2(B_1) & \xleftarrow{d_1} & H_2(B_2) & \xleftarrow{d_1} & H_2(B_3) & & \cdots \\
H_1(B_0) & \xleftarrow{d_1} & H_1(B_1) & \xleftarrow{d_1} & H_1(B_2) & \xleftarrow{d_1} & H_1(B_3) & & \cdots \\
H_0(B_0) & \xleftarrow{d_1} & H_0(B_1) & \xleftarrow{d_1} & H_0(B_2) & \xleftarrow{d_1} & H_0(B_3) & & \cdots
\end{array}$$

Note: In general, $d_1 \neq \partial_1$ and $\partial_1^2 \neq 0$. However, $d_1 = (\partial_1)_*$. This is very seldom true for the higher order differentials [Boardman] [Hurtubise].

The Austin-Braam de Rham cochain complex ∼1995

Let B_i be the set of critical points of index i and $C^{i,j} = \Omega^j(B_i)$ the set of j-forms on B_i. Austin and Braam define maps

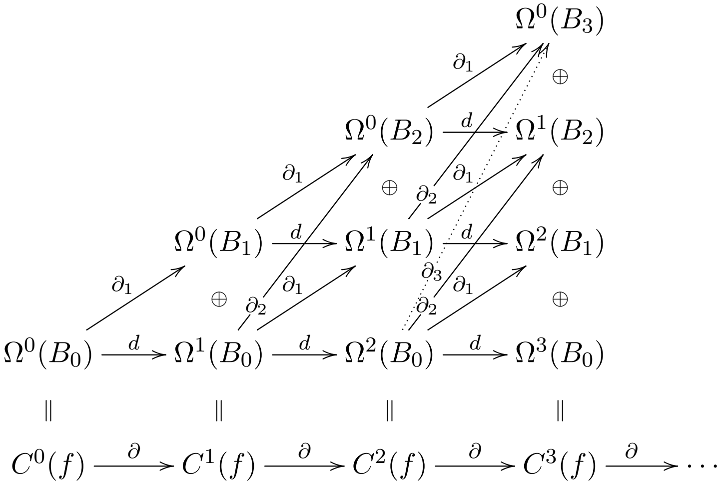
$$\partial_r : C^{i,j} \to C^{i+r,j-r+1}$$

for $r = 0, 1, 2, \ldots, m$ which raise the "total degree" $i + j$ by one. The map $\partial_0 = d$ and ∂_r is defined using integration along the fiber for $r = 1, 2, \ldots, m$. The maps $\partial_r : \Omega^j(B_i) \to \Omega^{j-r+1}(B_{i+r})$ fit together to form a cochain complex where $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ and

$$C^k(f) = \bigoplus_{i=0}^{k} \Omega^{k-i}(B_i).$$

Note: Integration along the fiber requires the "fibration condition" [Zhou], which is a consequence of M-B-S transversality.

The Austin-Braam M-B-S cochain complex



Other approaches

"abstract geometric chains" and fibered products
(Fukaya, Ruan, Tian) ∼1996

"collections of simplicial complexes"
(Liu and Tian) ∼1999

Kuranishi structures, fibered products, and spectral sequences
(Fukaya, Oh, Ohta, Ono) ∼2008

"abstract topological chains" and fibered products
(Banyaga Hurtubise) ∼2010

currents and homological perturbation theory
(Zhou) ∼2022

Moduli spaces of gradient flow lines

For any two critical submanifolds B and B' the flow φ_t induces an $\mathbb{R}$-action on $W^u(B) \cap W^s(B')$. Let

$$\mathcal{M}(B,B') = (W^u(B) \cap W^s(B'))/\mathbb{R}$$

be the quotient space of gradient flow lines from B to B'.

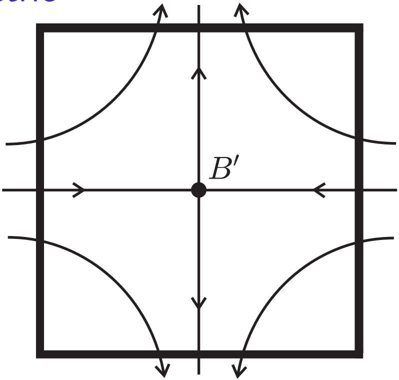
Theorem (Gluing)

Suppose that B, B', and B'' are critical submanifolds such that $W^u(B) \pitchfork W^s(B')$ and $W^u(B') \pitchfork W^s(B'')$. In addition, assume that $W^u(x) \pitchfork W^s(B'')$ for all $x \in B'$. Then for some $\epsilon > 0$, there is an injective local diffeomorphism

$$G : \mathcal{M}(B,B') \times_{B'} \mathcal{M}(B',B'') \times (0,\epsilon) \to \mathcal{M}(B,B'')$$

onto an end of $\mathcal{M}(B,B'')$. The gluing maps can be chosen to be associative.

The standard metric



$$G : \mathcal{M}(B,B') \times_{B'} \mathcal{M}(B',B'') \times (0,\epsilon) \to \mathcal{M}(B,B'')$$

The parameter $t \in (0,\epsilon)$ is related to the time to flow (or the distance) from $f^{-1}(f(B')+\varepsilon)$ to $f^{-1}(f(B')-\varepsilon)$.

Compactified moduli spaces

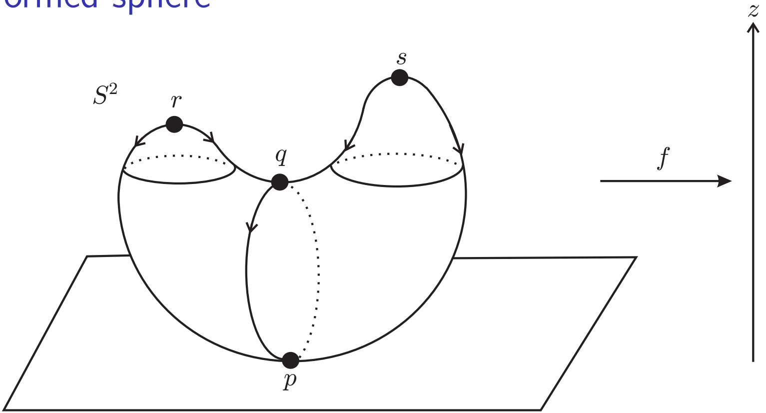
Theorem (Compactification)

Assume that $f : M \to \mathbb{R}$ satisfies the Morse-Bott-Smale transversality condition. For any two distinct critical submanifolds B and B' the moduli space $\mathcal{M}(B, B')$ has a compactification $\overline{\mathcal{M}}(B, B')$, consisting of all the piecewise gradient flow lines from B to B', which is a compact smooth manifold with corners of dimension $\lambda_B - \lambda_{B'} + b - 1$. Moreover, the beginning and endpoint maps extend to smooth maps

$$\begin{aligned} \partial_- &: \overline{\mathcal{M}}(B, B') \to B \\ \partial_+ &: \overline{\mathcal{M}}(B, B') \to B', \end{aligned}$$

where ∂_- has the structure of a locally trivial fiber bundle.

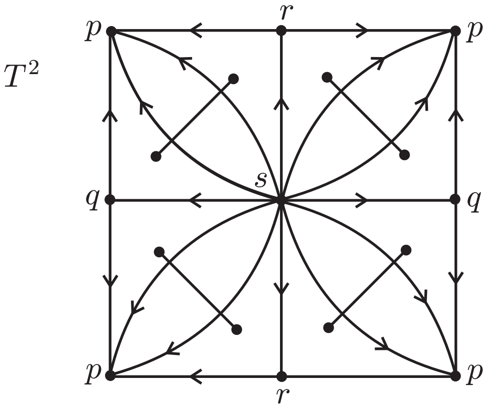
A deformed sphere



$$\mathcal{M}(r,p) \approx \mathcal{M}(s,p) \approx S^1 - \{pt\}$$

$$\overline{\mathcal{M}}(r,p) \approx \overline{\mathcal{M}}(s,p) \approx [0,1]$$

The flat torus (Diagram: R. Cohen)



$$\overline{\mathcal{M}}(s,p) \approx [0,1] \amalg [0,1] \amalg [0,1] \amalg [0,1]$$

The Austin-Braam coboundary operator

Pulling back along the endpoint map and then apply integration along the fiber using the beginning point map gives a "pull-push" operation that transports a differential form from B_i to B_{i+r}.

$$B_{i+r} \xleftarrow{\partial_-} \overline{\mathcal{M}}(B_{i+r}, B_i) \xrightarrow{\partial_+} B_i$$

Definition (Austin-Braam)

The map $\partial_r : \Omega^j(B_i) \to \Omega^{j-r+1}(B_{i+r})$ is defined by

$$\partial_r(\omega) = \begin{cases} d\omega & r = 0 \\ (-1)^j (\partial_-)_* (\partial_+^* \omega) & r \neq 0. \end{cases}$$

Integration along the fiber

Let $\pi : E \to B$ be a fiber bundle where B is a closed manifold, a typical fiber F is a compact oriented d-dimensional manifold with corners, and $\pi_\partial : \partial E \to B$ is also a fiber bundle with fiber ∂F. A differential form on E may be written locally as

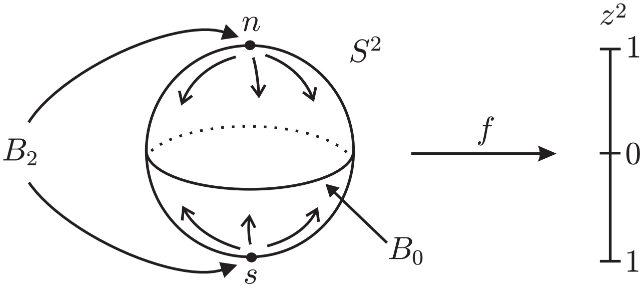
$$\pi^*(\phi) f(x,t)\, dt_{i_1} \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_r}$$

where ϕ is a form on B, x are coordinates on B, and the t_j are coordinates on F. Integration along the fiber $\pi_* : \Omega^j(E) \to \Omega^{j-d}(B)$ is defined by

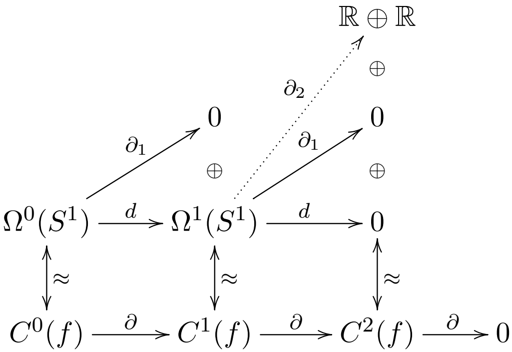
$$\begin{aligned} \pi_*(\pi^*(\phi) f(x,t)\, dt_1 \wedge dt_2 \wedge \cdots \wedge dt_d) &= \phi \int_F f(x,t)\, dt_1 \wedge \cdots \wedge dt_d \\ \pi_*(\pi^*(\phi) f(x,t)\, dt_{i_1} \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_r}) &= 0 \quad \text{if } r < d. \end{aligned}$$

A Morse-Bott-Smale function on S^2

Consider $S^2 = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}$, and let $f(x, y, z) = z^2$. Then $B_0 \approx S^1$, $B_1 = \emptyset$, and $B_2 = \{n, s\}$.



The Austin-Braam complex of $f(x,y,z) = z^2$



The second row computes the de Rham cohomology of S^1. Hence, $H^0(C^*(f), \partial) \approx \mathbb{R}$.

Orientations are required to define ∂_2

Note that $\overline{\mathcal{M}}(B_2, B_0) = \mathcal{M}(B_2, B_0) \approx S^1 \amalg S^1$, where the components have opposite orientations if $W^u(n)$ and $W^u(s)$ are given the same orientation as S^2. The map

$$(\partial_+)^* : \Omega^1(B_0) \to \Omega^1(\overline{\mathcal{M}}(B_2, B_0))$$

pulls back a 1-form ω to $\omega \amalg \omega \in \Omega^1(S^1) \amalg \Omega^1(S^1)$, and the map

$$(\partial_-)_* : \Omega^*(\overline{\mathcal{M}}(B_2, B_0)) \to \mathbb{R} \oplus \mathbb{R}$$

integrates a 1-form over the components.

$$\partial_2(\omega) = (-1)(\partial_-)_*(\partial_+^*\omega) = (c, \pm c),$$

$H^1(C^*(f), \partial) \approx 0$, and $H^2(C^*(f), \partial) \approx \mathbb{R}^2/\mathbb{R} \approx \mathbb{R}$.

Fibered products

Let $\sigma_i : P_i \to B$ for $i = 1, 2$ be two continuous maps into a topological space B. Recall that the fibered product of σ_1 and σ_2 is defined to be $P_1 \times_B P_2 = (\sigma_1 \times \sigma_2)^{-1}(\Delta)$, where Δ is the diagonal in $B \times B$, i.e.

$$P_1 \times_B P_2 = \{(x_1, x_2) \in P_1 \times P_2 | \ \sigma_1(x_1) = \sigma_2(x_2)\}.$$

Lemma

Suppose that $\sigma_1 : P_1 \to B$ and $\sigma_2 : P_2 \to B$ are smooth maps where P_1, P_2, and B are smooth manifolds (without boundary) of dimension p_1, p_2, and b respectively. If σ_1 is transverse to σ_2, then the fibered product $P_1 \times_B P_2$ is a smooth manifold of dimension $p_1 + p_2 - b$.

Proof: This follows from the fact that $\sigma_1 \pitchfork \sigma_2$ if and only if $(\sigma_1 \times \sigma_2) \pitchfork \Delta$.

Counterexample

Let $f : [-1,1] \to [-1,1] \times [-1,1]$ be given by

$$f(x) = \begin{cases} (x, e^{-1/x^2}\sin(\pi/x)) & \text{if } x \neq 0 \\ (0,0) & \text{if } x = 0 \end{cases}$$

and $g : [-1,1] \to [-1,1] \times [-1,1]$ be given by $g(x,y) = (x,0)$. Then f and g are smooth maps from finite dimensional compact oriented smooth manifolds with boundary whose fibered product $[-1,1] \times_{(f,g)} [-1,1] =$

$$\{(x,0) \in [-1,1] \times [-1,1] \mid x = 0, \pm 1, \pm 1/2, \pm 1/3, \ldots\}.$$

Hence, the fibered product of two finite CW-complexes might not be a CW-complex, and the fibered product of two finite simplicial complexes might not be a finite simplicial complex.

Theorem (L. Nielsen)

Let X and Y be C^s manifolds with corners, where $s \geq 1$. Let $A \subseteq Y$ be a C^s submanifold with corners, and $f : X \to Y$ a local C^s map, which preserves local facets relatively to A and intersects A transversally and stratum transversally. Then either $f^{-1}(A) = \emptyset$, or

1. *$f^{-1}(A)$ is a C^s submanifold with corners of X, and*
2. *dim X − dim $f^{-1}(A)$ = dim Y − dim A, and*
3. *ind(X, x) − ind$(f^{-1}(A), x)$ = ind$(Y, f(x))$ − ind$(A, f(x))$ for all $x \in f^{-1}(A)$.*

Note: When Y is a manifold without boundary the local facets condition is always satisfied.

Stratum transversality

The assumption that f intersects A **stratum transversally** means that for any $x \in f^{-1}(A)$ we have

$$df_x(\hat{T}_x X) + \hat{T}_y A = \hat{T}_y Y$$

where $y = f(x)$ and $\hat{T}_x X$ denotes the tangent space of the stratum containing $x \in X$. Similarly, we say that a map $f : X \to Y$ is a **stratum submersion** at $x \in X$ if and only if df_x maps $\hat{T}_x X$ onto $\hat{T}_y Y$ where $y = f(x)$.

Note that if f is a stratum submersion at $x \in X$ and $A \subseteq Y$ is any submanifold with corners containing y, then f intersects A stratum transversally.

Lemma (B-H)

For any two connected critical submanifolds B and B' of a Morse-Bott-Smale function, the beginning point map

$$\partial_- : \overline{\mathcal{M}}(B, B') \to B$$

is a submersion and a stratum submersion.

Corollary (B-H)

If B and B' are connected critical submanifolds of a Morse-Bott-Smale function and $\sigma : P \to B$ is a smooth map from a compact smooth manifold with corners P, then

$$P \times_B \overline{\mathcal{M}}(B, B')$$

is a compact smooth manifold with corners.

Triangulations and fibered products

Having **triangulations** on two spaces does not immediately induce a triangulation on the fibered product. In fact, there are simple diagrams of polyhedra and piecewise linear maps for which the diagram is **not triangulable**.

$$R \xleftarrow{g} P \xrightarrow{f} Q$$

There may not exist triangulations of P, Q, and R with respect to which both f and g are simplicial. [J.L. Bryant, *Triangulation and general position of PL diagrams*, Top. App. **34** (1990), 211-233]

The Banyaga-Hurtubise approach (∼2007)

Modeled on **cubical singular homology**. Based on ideas from Austin and Braam (∼1995), Barraud and Cornea (∼2004), Fukaya (∼1995), Weber (∼2006) etc.

Step 1: Generalize the notion of singular p-simplexes to allow maps from spaces other than the standard p-simplex $\triangle^p \subset \mathbb{R}^{p+1}$ or the unit p-cube $I^p \subset \mathbb{R}^p$. These generalizations of $\triangle^p$ (or I^p) are called **abstract topological chains**, and the corresponding singular chains are called **singular topological chains**.

Step 2: Show that ∂ extends to fibered products, and show that the compactified moduli spaces of gradient flow lines are abstract topological chains, i.e. ∂_0 is defined.

The Banyaga-Hurtubise approach (∼2007)

Step 3: Define the set of **allowed domains** C_p in the Morse-Bott-Smale chain complex as a collection of fibered products (with ∂_0 defined), and show that the allowed domains are all compact oriented smooth manifolds with corners.

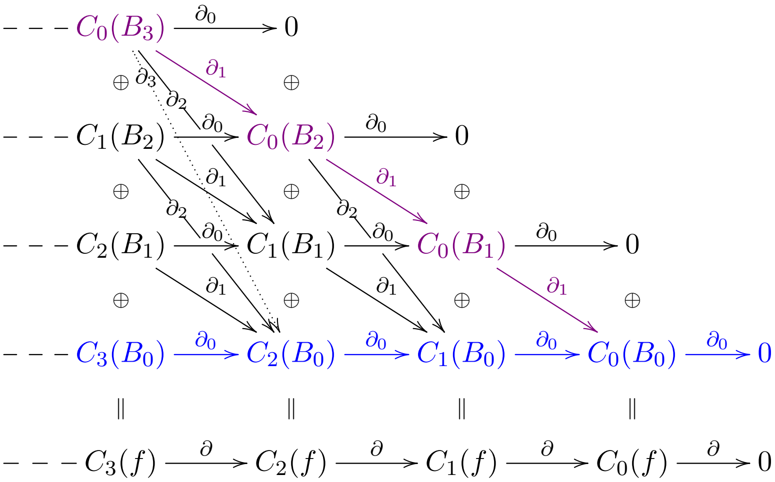
Step 4: Define $\partial_1, \ldots, \partial_m$ using fibered products of compactified moduli spaces of gradient flow lines and the beginning and endpoint maps. Define $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ and show that $\partial \circ \partial = 0$.

Step 5: Define **orientation conventions** on the elements of C_p and corresponding **degeneracy relations** to identify singular topological chains that are "essentially" the same. Show that $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ is compatible with the degeneracy relations.

The Banyaga-Hurtubise approach (∼2007)

Step 6: Show that the homology of the Morse-Bott-Smale chain complex $(C_*(f), \partial_*)$ is independent of $f : M \to \mathbb{R}$.
When $f : M \to \mathbb{R}$ is Morse-Smale, $(C_*(f), \partial_*)$ is the Morse-Smale-Witten complex, and when f is constant $(C_*(f), \partial_*)$ is the chain complex of cubical singular chains.

This gives a new proof of the Morse Homology Theorem which combines Morse chains and cubical singular chains in the same chain complex.


$C_0(B_3)$ ∂_0 0
$C_1(B_2)$ ∂_0 $C_0(B_2)$ ∂_0 0
$C_2(B_1)$ ∂_0 $C_1(B_1)$ ∂_0 $C_0(B_1)$ ∂_0 0
$C_3(B_0)$ ∂_0 $C_2(B_0)$ ∂_0 $C_1(B_0)$ ∂_0 $C_0(B_0)$ ∂_0 0
∂_1 ∂_2 ∂_3 $\oplus$ $\|$
$C_3(f)$ ∂ $C_2(f)$ ∂ $C_1(f)$ ∂ $C_0(f)$ ∂ 0

Step 1

For each integer $p \geq 0$ fix a set C_p of topological spaces, and let S_p be the free abelian group generated by the elements of C_p, i.e. $S_p = \mathbb{Z}[C_p]$. Set $S_p = \{0\}$ if $p < 0$ or $C_p = \emptyset$.

Definition

A **boundary operator** on the collection S_* of groups $\{S_p\}$ is a homomorphism $\partial_p : S_p \to S_{p-1}$ such that

1. For $p \geq 1$ and $P \in C_p \subseteq S_p$, $\partial_p(P) = \sum_k n_k P_k$ where $n_k = \pm 1$ and $P_k \in C_{p-1}$ is a subspace of P for all k.
2. $\partial_{p-1} \circ \partial_p : S_p \to S_{p-2}$ is zero.

We call (S_*, ∂_*) a **chain complex of abstract topological chains**. Elements of S_p are called **abstract topological chains** of **degree** p.

Step 1 continued

Definition

Let B be a topological space and $p \in \mathbb{Z}_+$. A **singular C_p-space** in B is a continuous map $\sigma : P \to B$ where $P \in C_p$, and the **singular C_p-chain group** $S_p(B)$ is the free abelian group generated by the singular C_p-spaces. Define $S_p(B) = \{0\}$ if $S_p = \{0\}$ or $B = \emptyset$. Elements of $S_p(B)$ are called **singular topological chains** of **degree** p.

Note: These definitions are quite general. To construct the M-B-S chain complex we really only need C_p to include the p-dimensional faces of an N-cube, the compactified moduli spaces of gradient flow lines of dimension p, and the components of their fibered products of dimension p.

Step 1 conclusion

For $p \geq 1$ there is a boundary operator $\partial_p : S_p(B) \to S_{p-1}(B)$ induced from the boundary operator $\partial_p : S_p \to S_{p-1}$. If $\sigma : P \to B$ is a singular C_p-space in B, then $\partial_p(\sigma)$ is given by the formula

$$\partial_p(\sigma) = \sum_k n_k \sigma|_{P_k}$$

where

$$\partial_p(P) = \sum_k n_k P_k.$$

The pair $(S_*(B), \partial_*)$ is called a **chain complex of singular topological chains** in B.

Abstract N-cube chains

Pick some large positive integer N and let

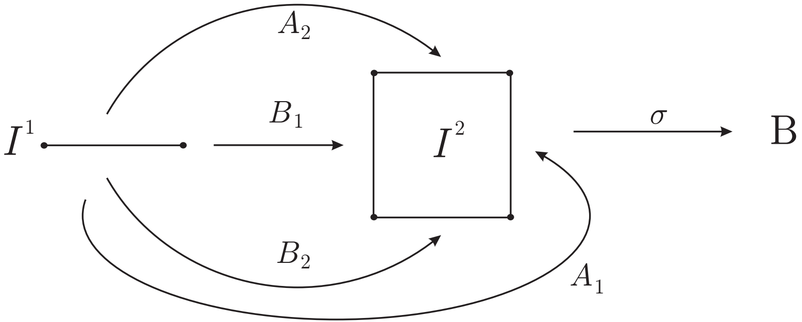
$$I^N = \{(x_1, \ldots, x_N) \in \mathbb{R}^N |\ 0 \leq x_j \leq 1,\ j = 1, \ldots, N\}$$

denote the unit N-cube. For every $0 \leq p \leq N$ let C_p be the set consisting of the faces of I^N of dimension p, i.e. subsets of I^N where p of the coordinates are free and the rest of the coordinates are fixed to be either 0 or 1. For every $0 \leq p \leq N$ let S_p be the free abelian group generated by the elements of C_p. For $P \in C_p$ define

$$\partial_p(P) = \sum_{j=1}^{p} (-1)^j \left[P|_{x_j=1} - P|_{x_j=0} \right] \in S_{p-1}$$

where x_j denotes the j^{th} free coordinate of P.

Cubical singular boundary operator (Massey)

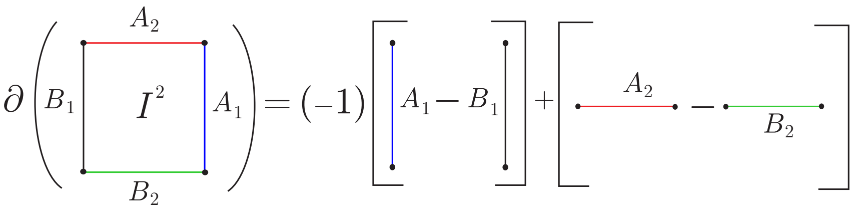


The chain $\sigma : I^2 \to B$ has boundary

$$\partial_2(\sigma) = (-1)[\sigma \circ A_1 - \sigma \circ B_1] + [\sigma \circ A_2 - \sigma \circ B_2]$$

where the terms in the sum are all maps with domain $I^1 = [0, 1]$.

Topological cubical boundary operator (B-H)



The chain $\sigma : I^2 \to B$ has boundary

$$\partial_2(\sigma) = (-1)[\sigma|_{A_1} - \sigma|_{B_1}] + [\sigma|_{A_2} - \sigma|_{B_2}]$$

and the degeneracy relations identify terms that are "essentially" the same.

Singular N-cube chains

A continuous map $\sigma_P : P \to B$ from a p-face P of I^N into a topological space B is a **singular C_p-space** in B. The boundary operator applied to σ_P is

$$\partial_p(\sigma_P) = \sum_{j=1}^{p} (-1)^j \left[\sigma_P|_{x_j=1} - \sigma_P|_{x_j=0}\right] \in S_{p-1}(B)$$

where $\sigma_P|_{x_j=0}$ denotes the restriction $\sigma_P : P|_{x_j=0} \to B$ and $\sigma_P|_{x_j=1}$ denotes the restriction $\sigma_P : P|_{x_j=1} \to B$.

Degeneracy relations

Definition

Let σ_P and σ_Q be singular C_p-spaces in B and let $\partial_p(Q) = \sum_j n_j Q_j \in S_{p-1}$. For any map $\alpha : P \to Q$, let $\partial_p(\sigma_Q) \circ \alpha$ denote the formal sum $\sum_j n_j (\sigma_Q \circ \alpha)|_{\alpha^{-1}(Q_j)}$. Define the subgroup $D_p(B) \subseteq S_p(B)$ of **degenerate singular N-cube chains** to be the subgroup generated by the following elements.

1. If α is an orientation preserving homeomorphism such that $\sigma_Q \circ \alpha = \sigma_P$ and $\partial_p(\sigma_Q) \circ \alpha = \partial_p(\sigma_P)$, then $\sigma_P - \sigma_Q \in D_p(B)$.
2. If σ_P does not depend on some free coordinate of P, then $\sigma_P \in D_p(B)$.

Theorem

The boundary operator for singular N-cube chains $\partial_p : S_p(B) \to S_{p-1}(B)$ descends to a homomorphism

$$\partial_p : S_p(B)/D_p(B) \to S_{p-1}(B)/D_{p-1}(B),$$

and

$$H_p(S_*(B)/D_*(B), \partial_*) \approx H_p(B; \mathbb{Z})$$

for all $p < N$.

Step 2

Show that ∂ extends to fibered products, and show that the compactified moduli spaces of gradient flow lines are abstract topological chains, i.e. ∂_0 is defined.

Fibered products

Suppose that $\sigma_1 : P_1 \to B$ is a singular S_{p_1}-space and $\sigma_2 : P_2 \to B$ is a singular S_{p_2}-space, where (S_*, ∂_*) is a chain complex of abstract topological chains. The **fibered product** of σ_1 and σ_2 is

$$P_1 \times_B P_2 = \{(x_1, x_2) \in P_1 \times P_2 | \ \sigma_1(x_1) = \sigma_2(x_2)\}.$$

This construction extends linearly to singular topological chains. The **degree** of the fibered product $P_1 \times_B P_2$ is defined to be $p_1 + p_2 - b$.

The **boundary operator** applied to the fibered product is defined to be

$$\partial(P_1 \times_B P_2) = \partial P_1 \times_B P_2 + (-1)^{p_1+b} P_1 \times_B \partial P_2$$

where ∂P_1 and ∂P_2 denote the boundary operator applied to the abstract topological chains P_1 and P_2. If $\sigma_1 = 0$, then we define $0 \times_B P_2 = 0$. Similarly, if $\sigma_2 = 0$, then $P_1 \times_B 0 = 0$.

Lemma

The fibered product of two singular topological chains is an abstract topological chain, i.e. the boundary operator on fibered products is of degree -1 and satisfies $\partial \circ \partial = 0$. Moreover, the boundary operator on fibered products is associative, i.e.

$$\partial((P_1 \times_{B_1} P_2) \times_{B_2} P_3) = \partial(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)).$$

Proof that $P_1 \times_B P_2$ is an abstract topological chain

Recall that the degree of $P_1 \times_B P_2$ is $p_1 + p_2 - b$.

Since ∂ is a boundary operator on P_1 and P_2, the degree of ∂P_1 is $p_1 - 1$ and the degree of ∂P_2 is $p_2 - 1$. Hence both $\partial P_1 \times_B P_2$ and $P_1 \times_B \partial P_2$ have degree $p_1 + p_2 - b - 1$.
To see that $\partial^2(P_1 \times_B P_2) = 0$ we compute as follows.

$$\begin{aligned} \partial(\partial(P_1 \times_B P_2)) &= \partial(\partial P_1 \times_B P_2 + (-1)^{p_1+b} P_1 \times_B \partial P_2) \\ &= \partial^2 P_1 \times_B P_2 + (-1)^{p_1-1+b} \partial P_1 \times_B \partial P_2 + \\ &\qquad (-1)^{p_1+b}(\partial P_1 \times_B \partial P_2 + (-1)^{p_1+b} P_1 \times_B \partial^2 P_2) \\ &= 0. \end{aligned}$$

Proof of associativity

Given the data of a triple

$$P_1 \xrightarrow{\sigma_{11}} B_1 \xleftarrow{\sigma_{12}} P_2 \xrightarrow{\sigma_{22}} B_2 \xleftarrow{\sigma_{23}} P_3$$

we can form the iterated fibered product

$$(P_1 \times_{B_1} P_2) \times_{B_2} P_3$$

using σ_{23} and the map $\sigma_{22} \circ \pi_2 : P_1 \times_{B_1} P_2 \to B_2$, where $\pi_2 : P_1 \times_{B_1} P_2 \to P_2$ denotes projection to the second component. Similarly, we can form the iterated fibered product

$$P_1 \times_{B_1} (P_2 \times_{B_2} P_3)$$

using σ_{11} and the map $\sigma_{12} \circ \pi_1 : P_1 \times_{B_1} P_2 \to B_1$, where $\pi_1 : P_2 \times_{B_2} P_3 \to P_2$ denotes projection to the first component.

$$\partial(P_1 \times_{B_1} (P_2 \times_{B_2} P_3))$$

$$\begin{aligned}
&= \partial P_1 \times_{B_1} (P_2 \times_{B_2} P_3) + (-1)^{p_1+b_1} P_1 \times_{B_1} \partial(P_2 \times_{B_2} P_3) \\
&= \partial P_1 \times_{B_1} (P_2 \times_{B_2} P_3) + \\
&\quad (-1)^{p_1+b_1} (P_1 \times_{B_1} (\partial P_2 \times_{B_2} P_3 + (-1)^{p_2+b_2} P_2 \times_{B_2} \partial P_3)) \\
&= \partial P_1 \times_{B_1} P_2 \times_{B_2} P_3 + (-1)^{p_1+b_1} P_1 \times_{B_1} \partial P_2 \times_{B_2} P_3 + \\
&\quad (-1)^{p_1+p_2+b_1+b_2} P_1 \times_{B_1} P_2 \times_{B_2} \times \partial P_3
\end{aligned}$$

$$\partial((P_1 \times_{B_1} P_2) \times_{B_2} P_3)$$

$$\begin{aligned}
&= \partial(P_1 \times_{B_1} P_2) \times_{B_2} P_3 + (-1)^{\deg(P_1 \times_{B_1} P_2)+b_2} (P_1 \times_{B_1} P_2) \times_{B_2} \partial P_3 \\
&= (\partial P_1 \times_{B_1} P_2 + (-1)^{p_1+b_1} P_1 \times_{B_1} \partial P_2) \times_{B_2} P_3 + \\
&\quad (-1)^{p_1+p_2-b_1+b_2} P_1 \times_{B_1} P_2 \times_{B_2} \partial P_3 \\
&= \partial P_1 \times_{B_1} P_2 \times_{B_2} P_3 + (-1)^{p_1+b_1} P_1 \times_{B_1} \partial P_2 \times_{B_2} P_3 + \\
&\quad (-1)^{p_1+p_2-b_1+b_2} P_1 \times_{B_1} P_2 \times_{B_2} \times \partial P_3
\end{aligned}$$

Compactified moduli spaces as abstract topological chains

Definition

Let $f : M \to \mathbb{R}$ be a Morse-Bott-Smale function, and let B_i be the set of critical points of index i. For any $j = 1, \ldots, i$ we define the **degree** of $\overline{\mathcal{M}}(B_i, B_{i-j})$ to be $j + b_i - 1$ and the **boundary operator** to be

$$\partial\overline{\mathcal{M}}(B_i, B_{i-j}) = (-1)^{i+b_i} \sum_{i-j<n<i} \overline{\mathcal{M}}(B_i, B_n) \times_{B_n} \overline{\mathcal{M}}(B_n, B_{i-j})$$

where $b_i = \text{dim } B_i$ and the fibered product is taken over the beginning and endpoint maps ∂_- and ∂_+. If $B_n = \emptyset$, then $\overline{\mathcal{M}}(B_i, B_n) = \overline{\mathcal{M}}(B_n, B_{i-j}) = 0$.

Lemma

The degree and boundary operator for $\overline{\mathcal{M}}(B_i, B_{i-j})$ satisfy the axioms for abstract topological chains, i.e. the boundary operator on the compactified moduli spaces is of degree -1 and $\partial \circ \partial = 0$.

Proof: Let $d = \deg \overline{\mathcal{M}}(B_i, B_n) = i - n + b_i - 1$.

Then $\partial(\overline{\mathcal{M}}(B_i, B_n) \times_{B_n} \overline{\mathcal{M}}(B_n, B_{i-j}))$

$$= \partial\overline{\mathcal{M}}(B_i, B_n) \times_{B_n} \overline{\mathcal{M}}(B_n, B_{i-j}) + (-1)^{d+b_n}\overline{\mathcal{M}}(B_i, B_n) \times_{B_n} \partial\overline{\mathcal{M}}(B_n, B_{i-j})$$

$$= (-1)^{i+b_i} \sum_{n<s<i} \overline{\mathcal{M}}(B_i, B_s, B_n, B_{i-j}) + (-1)^{i+b_i-1} \sum_{i-j<t<n} \overline{\mathcal{M}}(B_i, B_n, B_t, B_{i-j})$$

Therefore, $\partial^2 \overline{\mathcal{M}}(B_i, B_{i-j})$

$$\begin{aligned}
&= (-1)^{i+b_i} \left[\sum_{i-j<n<i} \left((-1)^{i+b_i} \sum_{n<s<i} \overline{\mathcal{M}}(B_i, B_s, B_n, B_{i-j}) + \right.\right. \\
&\qquad \left.\left. (-1)^{i+b_i-1} \sum_{i-j<t<n} \overline{\mathcal{M}}(B_i, B_n, B_t, B_{i-j}) \right) \right] \\
&= (-1)^{i+b_i} \left[(-1)^{i+b_i} \sum_{i-j<n<s<i} \overline{\mathcal{M}}(B_i, B_s, B_n, B_{i-j}) + \right. \\
&\qquad \left. (-1)^{i+b_i-1} \sum_{i-j<t<n<i} \overline{\mathcal{M}}(B_i, B_n, B_t, B_{i-j}) \right] \\
&= 0
\end{aligned}$$

$\square$

Step 3

Define the set of **allowed domains** C_p in the Morse-Bott-Smale chain complex as a collection of fibered products (with ∂_0 defined), and show that the allowed domains are all compact oriented smooth manifolds with corners.

For any $p \geq 0$, let C_p be the set consisting of the faces of I^N of dimension p and the connected components of degree p of fibered products of the form

$$Q \times_{B_{i_1}} \overline{\mathcal{M}}(B_{i_1}, B_{i_2}) \times_{B_{i_2}} \overline{\mathcal{M}}(B_{i_2}, B_{i_3}) \times_{B_{i_3}} \cdots \times_{B_{i_{n-1}}} \overline{\mathcal{M}}(B_{i_{n-1}}, B_{i_n})$$

where $m \geq i_1 > i_2 > \cdots > i_n \geq 0$, Q is a face of I^N of dimension $q \leq p$, $\sigma : Q \to B_{i_1}$ is smooth, and the fibered products are taken with respect to σ and the beginning and endpoint maps.

Theorem

The elements of C_p are compact oriented smooth manifolds with corners, and there is a boundary operator

$$\partial : S_p \to S_{p-1}$$

where S_p is the free abelian group generated by the elements of C_p.

Let $S_p^\infty(B_i)$ denote the subgroup of the singular C_p-chain group $S_p(B_i)$ generated by smooth maps $\sigma : P \to B_i$ such that $\sigma = \partial_+ \circ \pi$ whenever $P \in C_p$ is a connected component of a fibered product, where π denotes projection onto the last component of the fibered product.

Define $\partial_0 : S_p^\infty(B_i) \to S_{p-1}^\infty(B_i)$ by $\partial_0 = (-1)^{p+i}\partial$.

Step 4

Define $\partial_1, \ldots, \partial_m$ using fibered products of compactified moduli spaces of gradient flow lines and the beginning and endpoint maps. Define $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ and show that $\partial \circ \partial = 0$.

If $\sigma : P \to B_i$ is a singular C_p-space in $S_p^\infty(B_i)$, then for any $j = 1, \ldots, i$ composing the projection map π_2 onto the second component of $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ with the endpoint map $\partial_+ : \overline{\mathcal{M}}(B_i, B_{i-j}) \to B_{i-j}$ gives a map

$$P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\pi_2} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\partial_+} B_{i-j}.$$

Restricting $\partial_+ \circ \pi_2$ to the connected components of the fibered product and adding these restrictions (with the sign determined by the orientation when the dimension of a component is zero) defines an element $\partial_j(\sigma) \in S_{p+j-1}^\infty(B_{i-j})$.

Lemma

If $\sigma : P \to B_i$ is a singular C_p-space in $S_p^\infty(B_i)$, then for any $j = 1, \ldots, i$ adding the components of $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ (with sign when the dimension of a component is zero) yields an abstract topological chain of degree $p + j - 1$. That is, we can identify

$$P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \in S_{p+j-1}.$$

Thus, for all $j = 1, \ldots, i$ there is an induced homomorphism

$$\partial_j : S_p^\infty(B_i) \to S_{p+j-1}^\infty(B_{i-j})$$

which decreases the Morse-Bott degree $p + i$ by 1.

Proposition

$$\sum_{q=0}^{j} \partial_q \partial_{j-q} = 0, \text{ for every } j = 0, \ldots, m.$$

Proof: When $q = 0$ we compute:

$$\partial_0(\partial_j(P))$$

$$\begin{aligned}
&= \partial_0 \left(P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})\right) \\
&= (-1)^{p+i-1} \left(\partial P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) + (-1)^{p+b_i} P \times_{B_i} \partial \overline{\mathcal{M}}(B_i, B_{i-j})\right) \\
&= (-1)^{p+i-1} \partial P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \; + \\
&\quad (-1)^{2p+2b_i+2i-1} \sum_{i-j<n<i} P \times_{B_i} \overline{\mathcal{M}}(B_i, B_n) \times_{B_n} \overline{\mathcal{M}}(B_n, B_{i-j})
\end{aligned}$$

If $1 \leq q \leq j-1$, then

$$\partial_q(\partial_{j-q}(P)) = P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j+q}) \times_{B_{i-j+q}} \overline{\mathcal{M}}(B_{i-j+q}, B_{i-j})$$

and if $q = j$, then

$$\partial_j(\partial_0(P)) = (-1)^{p+i} \partial P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}).$$

Summing these expressions gives the desired result.

□

Corollary

The pair $(\tilde{C}_*(f), \partial)$ is a chain complex, where

$$\tilde{C}_k(f) = \bigoplus_{i=0}^{m} S^{\infty}_{k-i}(B_i)$$

and $\partial = \partial_0 \oplus \cdots \oplus \partial_m$.

Step 5

Define **orientation conventions** on the elements of C_p and corresponding **degeneracy relations** to identify singular topological chains that are "essentially" the same. Show that $\partial = \partial_0 \oplus \cdots \oplus \partial_m$ is compatible with the degeneracy relations.

Orientation conventions

Assume that every critical submanifold B and every negative normal bundle $\nu_*^-(B)$ are oriented. For any $p \in B$, the relation

$$T_pM = T_pB \oplus \nu_p^+(B) \oplus \nu_p^-(B)$$

determines an orientation on $\nu_p^+(B)$. The stable and unstable manifolds are oriented by requiring that the injective immersions $E^+ : \nu_*^+(B) \to W^s(B)$ and $E^- : \nu_*^-(B) \to W^u(B)$ are orientation preserving.

If $N \subseteq M$ is an oriented submanifold, then the normal bundle of N is oriented by the relation $T_x(N) \oplus \nu_x(N) = T_x(M)$ for all $x \in N$. For any two connected critical submanifolds B and B', the orientation on $W(B, B') = W^u(B) \pitchfork W^s(B')$ is determined by the relation

$$T_x(M) = T_xW(B, B') \oplus \nu_x(W^s(B')) \oplus \nu_x(W^u(B))$$

for all $x \in W(B, B')$. Picking a non-critical value a between $f(B')$ and $f(B)$ we can identify $\mathcal{M}(B, B') = f^{-1}(a) \cap W(B, B')$. An orientation on $\mathcal{M}(B, B')$ is then determined by

$$T_xW(B, B') = \text{span}((-\nabla f)(x)) \oplus T_x\mathcal{M}(B, B')$$

for all $x \in f^{-1}(a) \cap W(B, B')$. This determines an orientation on the compact manifold with boundary $\overline{\mathcal{M}}(B, B')$.

Definition

Suppose that B is an oriented smooth manifold without boundary and P_1 and P_2 are oriented smooth manifolds with corners. If $\sigma_1 : P_1 \to B$ and $\sigma_2 : P_2 \to B$ are smooth maps that intersect transversally and stratum transversally, then the orientation on the smooth manifold with corners $P_1 \times_B P_2$ is defined by the relation

$$(-1)^{(\text{dim } B)(\text{dim } P_2)} T_*(P_1 \times_B P_2) \oplus (\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B))) = T_*(P_1 \times P_2),$$

where $\nu_*(\Delta(B))$ denotes the normal bundle of the diagonal in $B \times$B.

Lemma

The above orientation on fibered products of transverse intersections of smooth manifolds with corners is associative.

Degeneracy relations

Let $\sigma_P, \sigma_Q \in S_p^\infty(B_i)$ be singular C_p-spaces in B_i and let $\partial Q = \sum_j n_j Q_j \in S_{p-1}$. For any map $\alpha : P \to Q$, let

$$\partial_0 \sigma_Q \circ \alpha \stackrel{def}{=} (-1)^{p+i} \sum_j n_j (\sigma_Q \circ \alpha)|_{\alpha^{-1}(Q_j)}.$$

Define the subgroup $D_p^\infty(B_i) \subseteq S_p^\infty(B_i)$ of **degenerate singular topological chains** to be the subgroup generated by the following elements.

1. If α is an orientation preserving diffeomorphism such that $\sigma_Q \circ \alpha = \sigma_P$ and $\partial_0 \sigma_Q \circ \alpha = \partial_0 \sigma_P$, then $\sigma_P - \sigma_Q \in D_p^\infty(B_i)$.
2. If P is a face of I^N and σ_P does not depend on some free coordinate of P, then $\sigma_P \in D_p^\infty(B_i)$ and $\partial_j(\sigma_P) \in D_{p+j-1}^\infty(B_{i-j})$ for all $j = 1, \ldots, m$.
3. If P and Q are connected components of some fibered products and α is an orientation reversing map such that $\sigma_Q \circ \alpha = \sigma_P$ and $\partial_0 \sigma_Q \circ \alpha = \partial_0 \sigma_P$, then $\sigma_P + \sigma_Q \in D_p^\infty(B_i)$.
4. If Q is a face of I^N and R is a connected component of a fibered product

$$Q \times_{B_{i_1}} \overline{\mathcal{M}}(B_{i_1}, B_{i_2}) \times_{B_{i_2}} \overline{\mathcal{M}}(B_{i_2}, B_{i_3}) \times_{B_{i_3}} \cdots \times_{B_{i_{n-1}}} \overline{\mathcal{M}}(B_{i_{n-1}}, B_{i_n})$$

such that $\deg R > \dim B_{i_n}$, then $\sigma_R \in D_r^\infty(B_{i_n})$ and $\partial_j(\sigma_R) \in D_{r+j-1}^\infty(B_{i_n - j})$ for all $j = 0, \ldots, m$.

5. If $\sum_\alpha n_\alpha \sigma_\alpha \in S_*(R)$ is a smooth singular chain in a connected component R of a fibered product (as in (4)) that represents the fundamental class of R and

$$\sum_\alpha n_\alpha(\partial_0 \sigma_R \circ \sigma_\alpha) - \sum_\alpha n_\alpha \partial_0(\sigma_R \circ \sigma_\alpha)$$

is in the group generated by the elements satisfying one of the above conditions, then

$$\sigma_R - \sum_\alpha n_\alpha(\sigma_R \circ \sigma_\alpha) \in D_r^\infty(B_{i_n})$$

and

$$\partial_j \left(\sigma_R - \sum_\alpha n_\alpha(\sigma_R \circ \sigma_\alpha) \right) \in D_{r+j-1}^\infty(B_{i_n - j})$$

for all $j = 1, \ldots, m$.

Step 6

Show that the homology of the Morse-Bott-Smale chain complex $(C_*(f), \partial_*)$ is independent of $f : M \to \mathbb{R}$.

Given two Morse-Bott-Smale functions $f_1, f_2 : M \to \mathbb{R}$ we pick a smooth function $F_{21} : M \times \mathbb{R} \to \mathbb{R}$ meeting certain transversality requirements such that

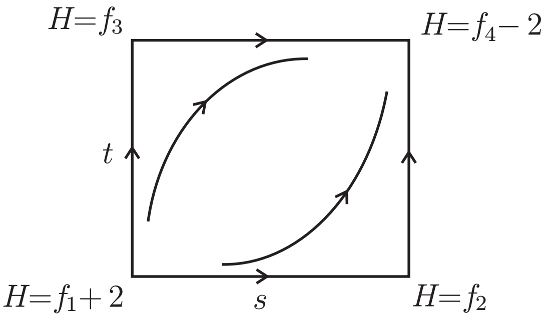
$$\begin{aligned} \lim_{t \to -\infty} F_{21}(x,t) &= f_1(x) + 1 \\ \lim_{t \to +\infty} F_{21}(x,t) &= f_2(x) - 1 \end{aligned}$$

for all $x \in M$. The compactified moduli spaces of gradient flow lines of F_{21} (the *time dependent* gradient flow lines) are used to define a chain map $(F_{21})_\square : C_*(f_1) \to C_*(f_2)$, where $(C_*(f_k), \partial)$ is the Morse-Bott chain complex of f_k for $k = 1, 2$.

Next we consider the case where we have four Morse-Bott-Smale functions $f_k : M \to \mathbb{R}$ where $k = 1, 2, 3, 4$, and we pick a smooth function $H : M \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ meeting certain transversality requirements such that

$$\begin{aligned}
\lim_{s\to-\infty}\lim_{t\to-\infty} H(x,s,t) &= f_1(x) + 2 \\
\lim_{s\to+\infty}\lim_{t\to-\infty} H(x,s,t) &= f_2(x) \\
\lim_{s\to-\infty}\lim_{t\to+\infty} H(x,s,t) &= f_3(x) \\
\lim_{s\to+\infty}\lim_{t\to+\infty} H(x,s,t) &= f_4(x) - 2
\end{aligned}$$

for all $x \in M$.


$H = f_3$
$H = f_4 - 2$
t
$H = f_1 + 2$
s
$H = f_2$

The compactified moduli spaces of gradient flow lines of H are used to define a chain homotopy between $(F_{43})_\square \circ (F_{31})_\square$ and $(F_{42})_\square \circ (F_{21})_\square$ where $(F_{lk})_\square : C_*(f_k) \to C_*(f_l)$ is the map defined above for $k, l = 1, 2, 3, 4$. In homology the map $(F_{kk})_* : H_*(C_*(f_k), \partial) \to H_*(C_*(f_k), \partial)$ is the identity for all k, and hence

$$\begin{aligned} (F_{12})_* \circ (F_{21})_* &= (F_{11})_* \circ (F_{11})_* = id \\ (F_{21})_* \circ (F_{12})_* &= (F_{22})_* \circ (F_{22})_* = id. \end{aligned}$$

Therefore,

$$(F_{21})_* : H_*(C_*(f_1), \partial) \to H_*(C_*(f_2), \partial)$$

is an isomorphism.

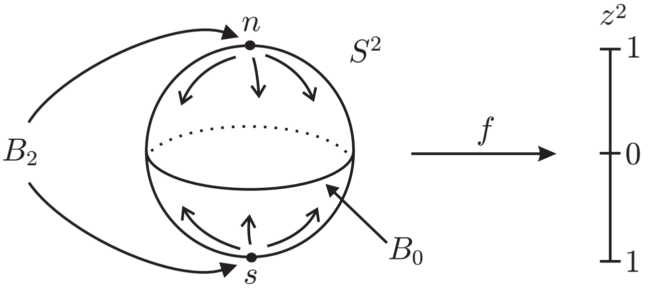
Theorem (Morse-Bott Homology Theorem)

The homology of the Morse-Bott chain complex $(C_*(f), \partial)$ *is independent of the Morse-Bott-Smale function* $f : M \to \mathbb{R}$. *Therefore,*

$$H_*(C_*(f), \partial) \approx H_*(M; \mathbb{Z}).$$

An example of Morse-Bott homology

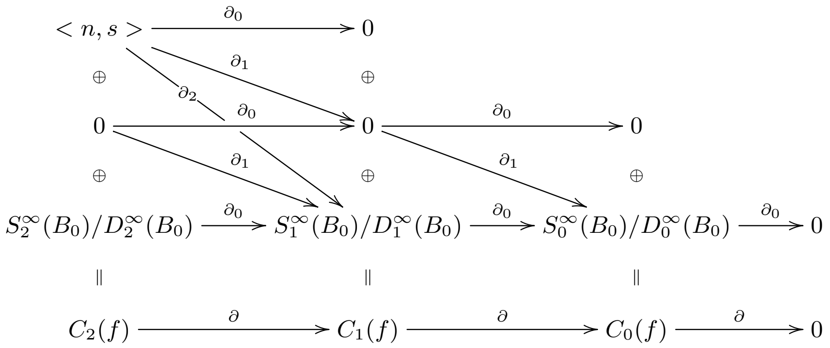
Consider $M = S^2 = \{(x, y, z) \in \mathbb{R}^3 | \; x^2 + y^2 + z^2 = 1\}$, and let $f(x, y, z) = z^2$. Then $B_0 \approx S^1$, $B_1 = \emptyset$, and $B_2 = \{n, s\}$.



The degeneracy conditions imply

$$S_0^\infty(B_2)/D_0^\infty(B_2) \approx < n, s > \approx \mathbb{Z} \oplus \mathbb{Z},$$

and $S_p^\infty(B_2)/D_p^\infty(B_2) = 0$ for $p > 0$.



The group $S_k^\infty(B_0)/D_k^\infty(B_0)$ is non-trivial for all $k \leq N$, but $H_k(C_*(f), \partial) = 0$ if $k > 2$ and

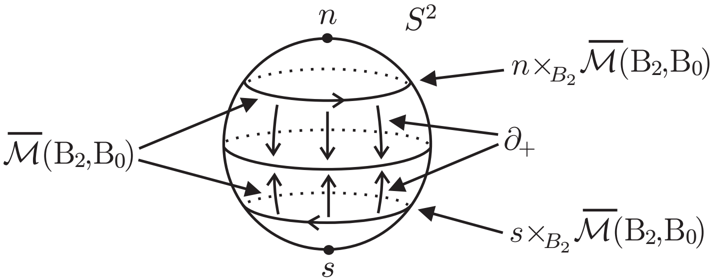
$$\partial_0 : S_3^\infty(B_0)/D_3^\infty(B_0) \to S_2^\infty(B_0)/D_2^\infty(B_0)$$

maps onto the kernel of the boundary operator

$$\partial_0 : S_2^\infty(B_0)/D_2^\infty(B_0) \to S_1^\infty(B_0)/D_1^\infty(B_0)$$

because the bottom row in the above diagram computes the smooth integral singular homology of $B_0 \approx S^1$.

The moduli space $\overline{\mathcal{M}}(B_2, B_0)$ is a disjoint union of two copies of S^1 with opposite orientations. This moduli space can be viewed as a subset of the manifold S^2 since $\overline{\mathcal{M}}(B_2, B_0) = \mathcal{M}(B_2, B_0)$.



There is an orientation reversing map

$$\alpha : n \times_n \overline{\mathcal{M}}(B_2, B_0) \to s \times_s \overline{\mathcal{M}}(B_2, B_0)$$

such that $\partial_2(n) \circ \alpha = \partial_2(s)$. Since $\partial_0(\partial_2(n)) = \partial_0(\partial_2(s)) = 0$, the degeneracy conditions imply that

$$\partial_2(n + s) = \partial_2(n) + \partial_2(s) = 0 \in S_1(B_0)/D_1(B_0).$$

They also imply that ∂_2 maps either n or s onto a representative of the generator of

$$\frac{\text{ker } \partial_0 : S_1^\infty(B_0)/D_1^\infty(B_0) \to S_0^\infty(B_0)/D_0^\infty(B_0)}{\text{im } \partial_0 : S_2^\infty(B_0)/D_2^\infty(B_0) \to S_1^\infty(B_0)/D_1^\infty(B_0)} \approx H_1(S^1;\mathbb{Z}) \approx \mathbb{Z}$$

depending on the orientation chosen for B_0. Therefore,

$$H_k(C_*(f), \partial) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

References

- David Austin and Peter Braam, *Morse-Bott theory and equivariant cohomology*, **The Floer memorial volume**, Progr. Math. **133** (1995), 123–183.
- David Austin and Peter Braam, *Equivariant Floer theory and gluing Donaldson polynomials*, Topology **35** (1996), no. 1, 167–200.
- Augustin Banyaga and David Hurtubise, *A proof of the Morse-Bott Lemma*, Expo. Math. **22**, no. 4, 365–373, 2004.
- Augustin Banyaga and David Hurtubise, **Lectures on Morse homology**, Kluwer Texts in the Mathematical Sciences **29**, Springer 2004.
- Augustin Banyaga and David Hurtubise, *The Morse-Bott inequalities via a dynamical systems approach*, Ergodic Theory Dynam. Systems **29** (2009), no. 6, 1693–1703.

References

- Augustin Banyaga and David Hurtubise, *Morse-Bott homology*, Trans. Amer. Math. Soc. **362** (2010), no. 8, 3997–4043.
- Augustin Banyaga and David Hurtubise, *Cascades and perturbed Morse-Bott functions*, Algebr. Geom. Topol. **13** (2013), 237-275.
- J. Michael Boardman, *Conditionally convergent spectral sequences*, Homotopy Invariant Algebraic Structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 49–84.
- Frédéric Bourgeois and Alexandru Oancea, *Symplectic homology, autonomous Hamiltonians, and Morse-Bott moduli spaces*, Duke Math. J. **146** (2009), no. 1, 74–174.

References

- Frédéric Bourgeois and Alexandru Oancea, *An exact sequence for contact and symplectic homology*, Invent. Math. **179** (2009), no. 3, 611–680.
- Andreas Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988), no. 2, 215–240.
- Andreas Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988), no. 2, 215–240.
- Andreas Floer, *Instanton homology, surgery, and knots*, **Geometry of low-dimensional manifolds**, London Math. Soc. Lecture Note Ser. **150** (1990), 97–114.
- Andreas Floer, *Morse theory for Lagrangian intersections*, Journal of Differential Geom **28** (1988), 513–547.

References

- Andreas Floer, *Witten's complex and infinite-dimensional Morse theory*, J. Differential Geom. **30** (1989), no. 1, 207–221.
- Urs Frauenfelder, *The Arnold-Givental conjecture and moment Floer homology*, Int. Math. Res. Not. **2004**, no. 42, 2179–2269.
- Kenji Fukaya, *Floer homology of connected sum of homology 3-spheres*, Topology **35** (1996), no. 1, 89–136.
- David Hurtubise, *Multicomplexes and spectral sequences*, J. Algebra Appl. **9** (2010), no. 4, 519–530.
- David Hurtubise, *Three approaches to Morse-Bott homology*, Afr. Diaspora J. Math. **14** (2012), no. 2, 145–177.
- Jean-Pierre Meyer, *Acyclic models for multicomplexes*, Duke Math. J. **45** (1978), no. 1, 67–85.

References

- Liviu Nicolaescu, **An Invitation to Morse Theory**, *Universitext*, Springer 2007.
- Yongbin Ruan and Gang Tian, *Bott-type symplectic Floer cohomology and its multiplication structures*, Math. Res. Lett. **2** (1995), no. 2, 203–219.
- Jan Swoboda, *Morse homology for the Yang-Mills gradient flow*, arXiv:1103.0845v1, 2011.
- Gang Liu and Gang Tian, *On the equivalence of multiplicative structures in Floer homology and quantum homology*, Acta Math. Sin. **15** (1999), no. 1, 53–80.
- Joa Weber, *The Morse-Witten complex via dynamical systems*, Expo. Math. **24** (2006), no. 2, 127–159.

References

- Edward Witten. **Supersymmetry and Morse theory**, J. Differential Geom. **17** (1982), no. 4, 661–692 (1983).
- Zhengyi Zhou, *Morse-Bott cohomology from homological perturbation theory*, Alg. Geom. Top.